MAU34106 - Galois Theory

Practice Sheet 2

Trinity College Dublin

Course homepage

These problems are just for practice, to help you warm up for the homework, and get more familiar with the material. I strongly encourage you to give them a try, as the best way to learn maths is through practice. They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems

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Exercise 1 *Abstract computations*

1. Show that $x^3 - 2x - 2$ is irreducible over $\mathbb{Q}$
2. Denote by α a root of of the polynomial. Express each of
$$\frac{1}{\alpha}, \quad \frac{1}{1+\alpha}, \quad \frac{1}{1+\alpha^2}$$
in the form
$$a + b\alpha + c\alpha^2, \quad a, b, c \in \mathbb{Q}.$$

Exercise 2 *Finding splitting fields*

1. Compute the degree of the splitting field of $x^4 - 6$ over $\mathbb{Q}$ and give a nice $\mathbb{Q}$-basis for the splitting field.

2. Compute the degree of the splitting field of $x^{12}-1$ over $\mathbb{Q}$ and give a nice $\mathbb{Q}$-basis.

3. Show that the splitting fields of $x^{12}-1$ and $(x^4-1)(x^3-1)$ coincide.

Exercise 3 *Modelling* $\mathbb{F}_8$

Let $K = \mathbb{F}_2[x]/(x^3+x^2+1)$ and $L = \mathbb{F}_2[y]/(y^3+y+1)$.

1. Show that both K and L are fields.

2. Determine the number of elements of K and L.

3. By giving an explicit map, show that $K \cong L$

   Hint: What is the minimal polynomial of $x+1$ over $\mathbb{F}_2$.

Exercise 4 *More splitting fields*

Let $f(x) \in K[x]$ be a polynomial of degree n, and let L/K be its splitting field. Show that

$$[L : K] \leq n!$$

Exercise 5 *Complexity*

Let $f(x) = x^3+x+1 \in \mathbb{Q}[x]$. You may assume freely that this is irreducible, and let K be the field $\mathbb{Q}[x]/(f(x))$

1. Is $K/\mathbb{Q}$ a separable extension? Why?

2. Is $K/\mathbb{Q}$ a normal extension? Why

   Hint: $f(x)$ is strictly increasing. Given any root $\alpha \in \mathbb{C}$, $K \cong \mathbb{Q}(\alpha)$. So?

3. Is $K/\mathbb{Q}$ a Galois extension? Why?

Exercise 6 *The real Galois group*

Determine $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ (by which I mean tell me what group it is isomorphic to).