MAU34106 - Galois Theory

Practice Sheet 1

Trinity College Dublin

Course homepage

These problems are just for practice, to help you warm up for the homework, and get more familiar with the material. I strongly encourage you to give them a try, as the best way to learn maths is through practice. They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems

---

Exercise 1 *Irreducible polynomials of low degree*

1. Let K be a field. Show that a polynomial of degree 2 or 3 is irreducible in $K[x]$ if and only if does not have a root in K.

2. Pick a field K and give an example of a degree 4 polynomial that is not irreducible, but does not have a root in K.

3. Show that $x^2 + x + 1$ is irreducible in $\mathbb{F}_2[x]$.

4. Show that $x^2 + 1$ is irreducible in $\mathbb{F}_3[x]$.

5. Show that $x^3 - 2x - 2$ is irreducible in $\mathbb{Q}[x]$

   Hint: Recall Gauss' Lemma: a polynomial with integer coefficients is irreducible in $\mathbb{Q}[x]$ *if and only if it is irreducible in* $\mathbb{Z}[x]$

Exercise 2 *Factorisation in finite fields*

Recall the finite field of order 2 $\mathbb{F}_2 = \{0, 1\}$. Write down every polynomial of degree exactly 4 as a product of irreducibles.

Remark: Although we will not present this method in the solution, it is arguably faster to write down every irreducible polynomial of degree at most 3*, and consider all possible products of these of degree* 4*. All polynomials not in this list of products will be irreducible*

Exercise 3 *Square roots and simple extensions*

1. Let K be a field of characteristic other than 2 and let L be an extension of degree 2. Show that $L = K(\sqrt{\alpha})$ for some $\alpha \in K$.

   Hint: Pick a K-basis $\{1, \beta\}$ of L, and note that the quadratic formula

   $$ax^2 + bx + c \quad \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   works in any field of characteristic other than 2.

2. We say that $c \in K$ is a square if there exists $d \in K$ such that $c = d^2$. Choose non-zero $a, b \in K$, and pick square roots $\sqrt{a}, \sqrt{b}$ in some field extension of K. Show that $K(\sqrt{a}) = K(\sqrt{b})$ if and only if $\frac{a}{b}$ is a square in K.

Exercise 4 *Degree calculations and minimal polynomials*

1. Show that
   $$[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$$

2. Show that
   $$[\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{6}) : \mathbb{Q}] < 8$$

3. Determine the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$. Be sure to justify minimality.

Exercise 5 *Classified*

Using the results of Exercise 3, classify all degree 2 extensions of $\mathbb{R}$.

Exercise 6 *Eisenstein won't help you here*

Show that $f(x) = x^{105} - 9$ is irreducible over $\mathbb{Z}$.

Hint: The roots of $f(x)$ are $\sqrt[105]{9}e^{\frac{2\pi ik}{105}}$ for $k = 0, 1, \ldots, 104$. Suppose we can factorise $f(x) = g(x)h(x)$ into polynomials with integer coefficients. What must those coefficients look like?

Exercise 7 *Hungry for power?*

Consider the first three symmetric power sums in $\mathbb{Q}[x_1, x_2, x_3]$:

$$\begin{aligned} h_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3, \\ h_2(x_1, x_2, x_3) &= x_1^2 + x_2^2 + x_3^2, \\ h_3(x_1, x_2, x_3) &= x_1^3 + x_2^3 + x_3^3. \end{aligned}$$

1. Write h_1, h_2, h_3 in terms of the elementary symmetric polynomials

$$\begin{aligned} e_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3, \\ e_2(x_1, x_2, x_3) &= x_1x_2 + x_1x_3 + x_2x_3, \\ e_3(x_1, x_2, x_3) &= x_1x_2x_3. \end{aligned}$$

2. Let $\alpha_1, \alpha_2, \alpha_3$ be the roots of

$$x^3 - 6x^2 + 21x + 15.$$

   Determine $\alpha_1^3 + \alpha_2^3 + \alpha_3^3$.

3. Determine a polynomial

$$x^3 + ax^2 + bx + c$$

   with roots $\beta_1, \beta_2, \beta_3$ such that

$$\begin{aligned} \beta_1 + \beta_2 + \beta_3 &= 2, \\ \beta_1^2 + \beta_2^2 + \beta_3^2 &= 42, \\ \beta_1^3 + \beta_2^3 + \beta_3^3 &= 62. \end{aligned}$$

   Hint: Try writing e_k in terms of h_ℓ.

4. Hence or otherwise solve

$$
\begin{aligned}
\beta_1 + \beta_2 + \beta_3 &= 2, \\
\beta_1^2 + \beta_2^2 + \beta_3^2 &= 42, \\
\beta_1^3 + \beta_2^3 + \beta_3^3 &= 62.
\end{aligned}
$$