MAU34106 - Galois Theory

Exercise Sheet 4

Trinity College Dublin

Course homepage

Answers are due for Friday, April 11^{th}, 17:00.
The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Please don't compute the discriminants by hand (100pt)*

Determine the Galois groups of the splitting fields of the following polynomials. You may use without proof the following results. Note that not every polynomial is necessarily irreducible.

- For a cubic polynomial $f(x) = x^3 + ax^2 + bx + c$, the discriminant is given by
$$a^2b^2 - 4b^3 - 4a^3c - 27c^2 + 18abc.$$

- For a quartic polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$, the discriminant is given by
$$\begin{aligned} 256d^3 &- 192acd^2 - 128b^2d^2 + 144bc^2d - 27c^4 + 144a^2bd^2 \\ &- 6a^2c^2d - 80ab^2cd + 18abc^3 + 16b^4d - 4b^3c^2 \\ &- 27a^4d^2 + 18a^3bcd - 4a^3c^3 - 4a^2b^3d + a^2b^2c^2 \end{aligned}$$

You are also welcome to, and encouraged to, use a computer to determine discriminants.

1. (25pts) $x^3 - 4x + 6$
2. (25pts) $x^3 - x^2 + x - 1$
3. (25pts) $x^4 - 2x^3 + 2x^2 + 2$
4. (25pts) $x^4 - 4x + 2$

Hint: You shouldn't need to think about primes greater than 5.

This exercise is entirely optional. If submitted before the deadline, your continuous assessment will consist of the best 3/4 assignments submitted. No extensions will be given

Further exercises on this topic can be found on the course webpage, and, I strongly encourage you to give them a try, as the best way to learn maths is through practice.

They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems

Solution 1

1. This is irreducible, by Eisenstein's criterion for $p = 2$, so $\mathrm{Gal}(f)$ is a transitive subgroup of S_3. There are only two options: S_3 and A_3. The discriminant is -716, which is not a square in $\mathbb{Q}$ (as it is negative), and so $\mathrm{Gal}(f) \not\subset A_3$. Thus $\mathrm{Gal}(f) = S_3$.

2. This is not irreducible. It factorises as
$$x^3 - x^2 + x - 1 = (x - 1)(x^2 + 1).$$
As $x^2 + 1$ is irreducible over $\mathbb{Q}$, the splitting field of the polynomial is $\mathbb{Q}(i)$, where i is a root of $x^2 + 1$. Thus
$$\mathrm{Gal}(f) = \mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z}$$
(as $\mathbb{Q}(i)$ is a Galois extension of degree 2 and there is only one group of order 2).

3. The polynomial is irreducible by Eisenstein's criterion for $p = 2$, and has discriminant
$$3136 = 2^6 7^2$$
which is a square. Thus $\mathrm{Gal}(f)$ is a transitive subgroup of S_4 contained in A_4. Thus $\mathrm{Gal}(f)$ is either A_4 or V_4. Consider $f(x)$ modulo 3. It reduces to
$$\overline{f}(x) = x^4 + x^3 - x^2 - 1.$$
This has a root in $\mathbb{F}_3$ at $x = 1$, and so we can factorise this as:
$$\overline{f}(x) = (x-1)(x^3 - x^2 + x + 1)$$
It is easy to check that the cubic factor has no roots in $\mathbb{F}_3$ and so is irreducible. Hence $\mathrm{Gal}(f)$ contains a 3-cycle, and so must be equal to A_4.

4. The polynomial is irreducible by Eisenstein's criterion for $p = 2$, and has discriminant -4864, which is not a square in $\mathbb{Q}$. Hence $\mathrm{Gal}(f)$ is a transitive subgroup of S_4 not contained in A_4. If we consider the reduction of $f(x)$ mod 5 we find
$$\overline{f}(x) = (x-2)(x^3 + 2x^2 - x - 1)$$
by testing for a root. By testing for a root, we find the cubic factor is irreducible, and so $\mathrm{Gal}(f)$ contains a 3-cycle. Hence $\mathrm{Gal}(f) = S_4$.