MAU34106 - Galois Theory

Exercise Sheet 3

Trinity College Dublin

Course homepage

Answers are due for Friday, April 4$^{\text{th}}$, 17:00.
The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *A biquadratic (100pt)*

The goal of this exercise to compute the Galois group of the splitting field of the polynomial

$$x^4 - 2x^2 - 5$$

over $\mathbb{Q}$. You may freely use the following:

- The polynomial $x^4 - 2x^2 - 5$ is irreducible over $\mathbb{Q}$, with two real roots α and $-\alpha$, and two imaginary roots β and $-\beta$.
- The minimal polynomial of an imaginary number over a subfield of $\mathbb{R}$ is of degree at least 2.
- If γ is algebraic over K, then $[K(\gamma) : K]$ is equal to the degree of the minimal polynomial of γ over K.
- Groups of order 8 can be distinguished using the properties described below the exercise.

1. (10pts) Determine the minimal polynomial of β over $\mathbb{Q}(\alpha)$.

   Hint: Consider writing the coefficients of the polynomial in terms of the roots

2. (10pts) Hence, or otherwise, determine the degree of the splitting field $\mathbb{Q}(\alpha, \beta)/\mathbb{Q}$.

3. (30pts) Explain why the $G = \mathrm{Gal}(\mathbb{Q}(\alpha, \beta)/\mathbb{Q})$ is of order 8. By explicitly describing their action on α and β, write down all elements of G.

   Hint: Recall that $\sigma \in G$ must permute the roots of $x^4 - 2x^2 - 5$. In particular, $\sigma(\alpha)$ and $\sigma(\beta)$ must be distinct roots of the polynomial

4. (20pts) Hence identify G with one of the five groups of order 8.

5. (30pts) Apply the Galois correspondence: identify the fixed subfield of a subgroup of order 4 and a subgroup of order 2.

6. (Optional, but good practice!) Using the Galois correspondence, identify all intermediate fields $\mathbb{Q} \subset F \subset \mathbb{Q}(\alpha, \beta)$. Which $F/\mathbb{Q}$ are Galois extensions?

These are the only exercises that you must submit before the deadline

Further exercises on this topic can be found on the course webpage, and, I strongly encourage you to give them a try, as the best way to learn maths is through practice.

They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems

Groups of order 8

There are 5 groups of order 8

$$\mathbb{Z}/8\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},\ (\mathbb{Z}/2\mathbb{Z})^2,\ D_4,\ Q_8$$

More details about these groups can be found here. These can be distinguished by checking whether they are abelian, and how many elements of each order there are:

- $\mathbb{Z}/8\mathbb{Z}$ is an abelian group with 4 elements of order 8, 2 of order 4, 1 of order 2, and the identity.
- $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is abelian, with 4 elements of order 4, 3 elements of order 2, and the identity.
- $(\mathbb{Z}/2\mathbb{Z})^3$ is abelian, with 7 elements of order 2, and the identity.
- D_4 is non-abelian, with 2 elements of order 4, 5 elements of order 2, and the identity.
- Q_8 is non-abelian, with 6 elements of order 4, 1 of order 2, and the identity.