MAU34106 - Galois Theory

Exercise Sheet 2

Trinity College Dublin

Course homepage

Answers are due for February 28^{th}, 17:00.
The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *A tale of two splitting fields (100pt)*

The goal of this exercise to to give an example of irreducible polynomials $f(x) \in \mathbb{Q}[x]$, such that the abstract field extension

$$K = \mathbb{Q}[x]/(f(x))$$

is or is not the splitting field of $f(x)$. It will be convenient to denote by α the image of x in the quotient, so we can think of K as a field of the form $\mathbb{Q}(\alpha)$ for some α such that $f(\alpha) = 0$.

1. (30pts) Consider the polynomial $f(x) = x^4 - 5x^2 + 1$, and let α denote the image of x in K. Show that $f(x)$ splits as a product of linear factors in $K[x]$

   Hint: What symmetries does $f(x)$ have? Can you find the roots in terms of α?

2. Consider the polynomial $f(x) = x^3 - 2$, and let α denote the image of x in K. We will show that K is not the splitting field of $f(x)$.

   a) (10pts) Verify that

   $$f(x) = (x - \alpha)(x^2 + \alpha x + \alpha^2)$$

   in $K[x]$.

b) (30pts) To show that $x^2 + \alpha x + \alpha^2$ is irreducible in $K[x]$, it suffices to show that it has no roots in K. Suppose otherwise that we have a root of the form

$$a + b\alpha + c\alpha^2, \quad a, b, c \in \mathbb{Q}.$$

Suppose that $a, c \neq 0$ and $2b \neq -1$, and show that

$$\left(\frac{a}{c}\right)^3 = 4$$

Hint: Recall $\alpha^3 = 2$ and that $1, \alpha, \alpha^2$ forms a $\mathbb{Q}$-basis

c) (30pts) Hence or otherwise, conclude that $x^2 + \alpha x + \alpha^2$ has no roots in K.

Hint: You may use that $x^3 - 4$ and $x^2 + x + 1$ have no rational roots!

These are the only exercises that you must submit before the deadline

Further exercises on this topic can be found on the course webpage, and, I strongly encourage you to give them a try, as the best way to learn maths is through practice.

They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems

Solution 1

1. Note that $f(x)$ is even, and its coefficients form a palindrome. Hence, we have that

$$f(-x) = f(x) \quad \text{and} \quad f(\frac{1}{x}) = \frac{1}{x^4} f(x)$$

As such, if $f(\alpha) = 0$

$$f(-\alpha) = f\left(\frac{1}{\alpha}\right) = f\left(\frac{-1}{\alpha}\right) = 0$$

Thus, $\alpha, -\alpha, \frac{1}{\alpha}, \frac{-1}{\alpha}$ are all roots of $f(x)$, and are all contained in K. As $f(x)$ has degree 4, if these are all distinct, this implies $f(x)$ splits as a product of linear factors in $K[x]$.

Clearly $\alpha \neq -\alpha$, as 0 is not a root of $f(x)$. Similarly, as ± 1 are not roots of $f(x)$, $\alpha \neq \frac{1}{\alpha}$. Finally, $\alpha \neq -\frac{1}{\alpha}$, as $\alpha^2 = -1$ does not give a root of $f(x)$. This ensure that we get four distinct roots.

2. a) We know that $\alpha^3 = 2$ in K, so it is just a matter of multiplying this out

$$\begin{aligned}(x-\alpha)(x^2+\alpha x+\alpha^2) &= x3 - \alpha x^2 + \alpha x^2 - \alpha^2 x + \alpha^2 x - \alpha^3 \\ &= x^3 - \alpha^3 = x^3 - 2.\end{aligned}$$

   b) Suppose we have such a root. Then

$$(a+b\alpha+c\alpha^2)^2 + \alpha(a+b\alpha+c\alpha^2) + \alpha^2 = 0$$

and hence

$$(b^2+2ac+b+1)\alpha^2 + (2c^2+2ab+a)\alpha + (a^2+4bc+2c) = 0,$$

where we have used that $\alpha^3 = 2$ and $\alpha^4 = 2\alpha$. As $1, \alpha, \alpha^2$ form a basis, we must have that

$$\begin{aligned}b^2+2ac+b+1 &= 0,\\ 2c^2+a(2b+1) &= 0,\\ a^2+2c(2b+1) &= 0.\end{aligned}$$

Rearranging the last two of these, we find

$$2c^2 = -a(2b+1), \quad a^2 = -2c(2b+1).$$

Dividing one of these by the other (which we can do from the assumptions on a, b, c), we get

$$\frac{a^2}{2c^2} = \frac{2c}{a} \quad \Rightarrow \frac{a^3}{c^3} = 4$$

as required.

c) If $a \neq 0$, $b \neq \frac{-1}{2}$ and $c \neq 0$, then we must have that

$$\left(\frac{a}{c}\right)^3 = 4$$

and so $\frac{a}{c}$ is a rational root of $x^3 - 4$. But no such root exists, so this is impossible.

If $a = 0$, then the second equality from the previous part implies that $c = 0$ and hence

$$b^2 + b + 1 = 0.$$

But $x^2 + x + 1$ has no rational roots, so this is impossible.

We get a similar conclusion if $c = 0$. Finally, if $b = \frac{-1}{2}$, then we find $a = c = 0$, and so $b^2 + b + 1 = 0$, which cannot hold. Thus, no such a, b, c can exist and $x^2 + \alpha x + \alpha^2$ has no roots in K.