MAU34106 - Galois Theory

Exercise Sheet 1

Trinity College Dublin

Course homepage

Answers are due for September 28th, 2pm.
The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Power sum polynomials (100pt)*

Let K be a field of characteristic 0. In class, we sketched an argument that every symmetric polynomial could be written in terms of elementary symmetric polynomials. It is also true that every symmetric polynomial can be written in terms of the power sums

$$h_k(x_1, \ldots, x_n) = x_1^k + x_2^k + \cdots + x_n^k$$

for $1 \leq k \leq n$. The goal of this problem is to prove this for $n = 4$

1. (10pts) Write down $e_1(x_1, x_2, x_3, x_4)$, $e_2(x_1, x_2, x_3, x_4)$, $e_3(x_1, x_2, x_3, x_4)$, and $e_4(x_1, x_2, x_3, x_4)$
2. (10pts) Clearly $h_1 = e_1$. By considering all products of degree 2 (or otherwise), express h_2 as a polynomial in e_1, e_2, e_3, e_4.
3. (30pts) By considering all products of degrees 3 and 4 (or otherwise), express h_3 and h_4 as polynomials in e_1, e_2, e_3, e_4.
4. (20pts) Solve these polynomial equations to write e_1, e_2, e_3 and e_4 as polynomials in h_1, h_2, h_3, h_4.

5. (30pts) Hence or otherwise, find a polynomial

$$f(x) = x^4 + ax^3 + bx^2 + cx + d$$

whose roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfy

$$\begin{aligned}
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 1 \\
\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 &= 2 \\
\alpha_1^3 + \alpha_2^3 + \alpha_3^3 + \alpha_4^3 &= 3 \\
\alpha_1^4 + \alpha_2^4 + \alpha_3^4 + \alpha_4^4 &= 4
\end{aligned}$$

Be careful with signs!

These are the only exercises that you must submit before the deadline

Further exercises on this topic can be found on the course webpage, and, I strongly encourage you to give them a try, as the best way to learn maths is through practice.

They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems

Solution 1

In these solutions, we will suppress the dependency on x_1, x_2, x_3, x_4 where it is clear from context

1.

$$\begin{aligned}
e_1 &= x_1 + x_2 + x_3 + x_4 \\
e_2 &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \\
e_3 &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 \\
e_4 &= x_1x_2x_3x_4
\end{aligned}$$

2. There are two possible products of elementary symmetric polynomials of degree 2: e_1^2 and e_2. We have that

$$e_1^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)$$

so $e_1^2 = h_2 + 2e_2$ and therefore

$$h_2 = e_1^2 - 2e_2$$

3. There are three possible products of elementary symmetric polynomials of degree 3: e_1^3, e_1e_2 and e_3. We find that

$$\begin{aligned}
e_1^3 =& x_1^3 + x_2^3 + x_3^3 + x_4^3 \\
&+ 3(x_1^2x_2 + x_1^2x_3 + x_1^2x_4 + x_2^2x_3 + x_2^2x_4 + x_3^2x_4) \\
&+ 3(x_1x_2^2 + x_1x_3^2 + x_1x_4^2 + x_2x_3^2 + x_2x_4^2 + x_3x_4^2) \\
&+ 6(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) \\
e_1e_2 =& x_1^2x_2 + x_1^2x_3 + x_1^2x_4 + x_2^2x_3 + x_2^2x_4 + x_3^2x_4 \\
&+ x_1x_2^2 + x_1x_3^2 + x_1x_4^2 + x_2x_3^2 + x_2x_4^2 + x_3x_4^2 \\
&+ 3(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) \\
e_3 =& x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4.
\end{aligned}$$

Comparing coefficients in

$$h_3 = ae_1^3 + be_1e_2 + ce_3$$

we find that

$$h_3 = e_1^3 - 3e_1e_2 + 3e_3$$

Similarly, we compute

$$\begin{aligned}
e_1^4 =& x_1^4 + x_2^4 + x_3^4 + x_4^4 \\
&+ 4x_1^3x_2 \ + \cdots + 6x_1^2x_2^2 + \cdots \\
&+ 12x_1^2x_2x_3 + 24x_1x_2x_3x_4 \\
e_1^2e_2 =& x_1^3x_2 + \cdots + 2x_1^2x_2^2 + \cdots \\
&+ 5x_1x_2x_3^2 + \cdots + 12x_1x_2x_3x_4 \\
e_2^2 =& x_1^2x_2^2 + \cdots + 2x_1x_2x_3^2 + \cdots + 6x_1x_2x_3x_4 \\
e_1e_3 =& x_1x_2x_3^2 + \cdots + 4x_1x_2x_3x_4
\end{aligned}$$

where we have only include one monomial in each symmetry class, for sake of trees. Writing

$$h_4 = ae_1^4 + e_1^2 e_2 + ce_2^2 + de_1 e_3 + qe_4$$

and comparing coefficients, we find

$$\begin{aligned} a &= 1 \\ 4a + b &= 0 \\ 6a + 2b + c &= 0 \\ 12a + 5b + 2c + d &= 0 \\ 24a + 12b + 6c + 4d + q &= 0. \end{aligned}$$

Hence

$$a = 1,\ b = -4,\ c = 2,\ d = 4,\ q = -4$$

and so $h_4 = e_1^4 - 4e_1^2 e_2 + 2e_2^2 + 4e_1 e_3 - 4e_4$.

4. Clearly $e_1 = h_1$. Thus

$$h_2 = e_1^2 - 2e_2 = h_1^2 - 2e_2$$

and

$$e_2 = \frac{1}{2}\left(h_1^2 - h_2\right).$$

Filling this into our expression for h_3, we get

$$h_3 = h_1^2 - \frac{3}{2} h_1 (h_1^2 - h_2) + 3e_3$$

and hence

$$e_3 = \frac{1}{6} h_1^3 - \frac{1}{2} h_1 h_2 + \frac{1}{3} h_3.$$

Finally

$$h_4 = h_1^4 - 2h_1^2 (h_1^2 - h_2) + 4h_1 \left(\frac{1}{6} h_1^3 - \frac{1}{2} h_1 h_2 + \frac{1}{3} h_3\right) + \frac{1}{2} (h_1^2 - h_2)^2 - 4e_4$$

and so

$$e_4 = \frac{1}{24} h_1^4 - \frac{1}{4} h_1^2 h_2 + \frac{1}{3} h_1 h_3 + \frac{1}{8} h_2^2 - \frac{1}{4} h_4$$

5. Recall that if $f(x)$ has roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, then

$$\begin{aligned} a &= -e_1(\alpha_1, \dots, \alpha_4), \\ b &= e_2(\alpha_1, \dots, \alpha_4), \\ c &= -e_3(\alpha_1, \dots, \alpha_4), \\ d &= e_4(\alpha_1, \dots, \alpha_4). \end{aligned}$$

From the question, we know that

$$h_k(\alpha_1, \dots, \alpha_4) = k$$

for $k = 1, \dots, 4$. Hence

$$\begin{aligned} e_1(\alpha_1, \dots, \alpha_4) = h_1 = 1&, \\ e_2 &= \frac{1}{2}(h_1^2 - h_2) = \frac{1}{2}(1-2) = -\frac{1}{2}, \\ e_3 &= \frac{1^3}{6} - \frac{1 \times 2}{2} + \frac{3}{3} = \frac{1}{6}, \\ e_4 &= \frac{1}{24} - \frac{2}{4} + \frac{3}{3} + \frac{4}{8} - \frac{4}{4} = \frac{1}{24}. \end{aligned}$$

Hence $\alpha_1, \dots, \alpha_4$ are the roots of

$$f(x) = x^4 - x^3 - \frac{1}{2}x^2 - \frac{1}{6}x + \frac{1}{24}$$