MAU34106 - Galois Theory

Exercise Sheet 1

Trinity College Dublin

Course homepage

Answers are due for February 14th, 17:00. The use of electronic calculators and computer algebra software is allowed, and for this problem, encouraged.

Exercise 1 Power sum polynomials (100pt)

Let K be a field of characteristic 0. In class, we sketched an argument that every symmetric polynomial could be written in terms of elementary symmetric polynomials. It is also true that every symmetric polynomial can be written in terms of the power sums

$$h_k(x_1,\ldots,x_n) = x_1^k + x_2^k + \cdots + x_n^k$$

for $1 \le k \le n$. The goal of this problem is to prove this for n = 4

- 1. (10pts) Write down $e_1(x_1, x_2, x_3, x_4)$, $e_2(x_1, x_2, x_3, x_4)$, $e_3(x_1, x_2, x_3, x_4)$, and $e_4(x_1, x_2, x_3, x_4)$
- 2. (10pts) Clearly $h_1 = e_1$. By considering all products of degree 2 (or otherwise), express h_2 as a polynomial in e_1, e_2, e_3, e_4 .
- 3. (30pts) By considering all products of degrees 3 and 4 (or otherwise), express h_3 and h_4 as polynomials in e_1, e_2, e_3, e_4 .
- 4. (20pts) Solve these polynomial equations to write e_1 , e_2 , e_3 and e_4 as polynomials in h_1 , h_2 , h_3 , h_4 .

5. (30pts) Hence or otherwise, find a polynomial

$$f(x) = x^4 + ax^3 + bx^2 + cx + d$$

whose roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfy

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$$

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = 2$$

$$\alpha_1^3 + \alpha_2^3 + \alpha_3^3 + \alpha_4^3 = 3$$

$$\alpha_1^4 + \alpha_2^4 + \alpha_3^4 + \alpha_4^4 = 4$$

Be careful with signs!

These are the only exercises that you must submit before the deadline

Further exercises on this topic can be found on the course webpage, and, I strongly encourage you to give them a try, as the best way to learn maths is through practice.

They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems