Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Science, Technology, Engineering and Mathematics

School of Mathematics

SF Mathematics **Michaelmas Term 2024**
SF Joint Honours
JS TP

Analysis in Several Real Variables - Sample Exam 2

Day **Place** **Time**

Dr. Adam Keilthy

Instructions to candidates:

Attempt any three questions. If you attempt all four questions, only your best three will be considered in your grade. All questions are worth 30 points

Unless stated otherwise, you may use all statements given lectures without proof, but must clearly justify that the assumptions of statement are fulfilled.

Additional instructions for this examination:

You may use a non-programmable calculator. Please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

Question 1

Let $M_{2\times 2}(\mathbb{R})$ denote the set of (2×2)-matrices, and let $\mathrm{GL}_2(\mathbb{R})$ denote the subset of invertible matrices.

i) (8pts) Determine an explicit isomorphism of real vector spaces $\Psi : \mathbb{R}^4 \to M_{2\times 2}$.

ii) (8pts) Show that the map $\mathbb{R}^4 \to \mathbb{R}$ given by

$$\vec{v} \mapsto \det \Psi(\vec{v})$$

is continuous on $\mathbb{R}^4$.

iii) (6pts) Hence determine whether $\Psi^{-1}(\mathrm{GL}_2(\mathbb{R}))$ is open or closed or neither as a subset of of $\mathbb{R}^4$.

iv) (8pts) Prove that the map

$$\vec{v} \mapsto \Psi^{-1}\left(\Psi(\vec{v})^{-1}\right)$$

induced by matrix inversion is continuous on $\Psi^{-1}(\mathrm{GL}_2(\mathbb{R}))$.

Question 2

1. (8pts) State, with all necessary hypotheses, the chain rule for differentiable functions of several real variables

2. (10pts) Let $g : \mathbb{R} \to \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}$ be functions differentiable everywhere. Using the chain rule, show that

$$(f \circ g)'(t) = \sum_{k=1}^{m} \frac{\partial f}{\partial x_k}(g(t))\frac{dg_k}{dt}(t)$$

for every $t \in \mathbb{R}$, where

$$g(t) = (g_1(t), \ldots, g_m(t)).$$

3. (12pts) Fix $(x_1, \ldots, x_m) \in \mathbb{R}^m$. Hence, or otherwise, show that there exists $\delta > 0$ and $\theta \in (0, 1)$ such that

$$\begin{aligned} &f(x_1 + a_1 h, x_2 + a_2 h, \ldots, x_m + a_m h) \\ &= f(x_1, \ldots, x_m) + h\sum_{k=1}^{m} a_k \frac{\partial f}{\partial x_k}(x_1 + a_1\theta h, x_2 + a_2\theta h, \ldots, x_m + a_m\theta h) \end{aligned}$$

for all $h \in (-\delta, \delta)$.

Question 3

Recall that a real symmetric $(m \times m)$-matrix A is called positive definite if $\langle \vec{x}, A\vec{x} \rangle > 0$ for all non-zero $\vec{x} \in \mathbb{R}^m$. Given a positive definite matrix A, we define the A-norm by

$$\|\vec{x}\|_A = \sqrt{\langle \vec{x}, A\vec{x} \rangle}.$$

1. (5pts) Show that every eigenvalue of A is positive.
2. (8pts) Hence show that the exist constants $c, C > 0$ such that

$$c\|\vec{x}\| \leq \|\vec{x}\|_A \leq C\|\vec{x}\|$$

   for all $\vec{x} \in \mathbb{R}^m$.
3. (10pts) Show that $\|\vec{x} + \vec{y}\|_A \leq \|\vec{x}\|_A + \|\vec{y}\|_A$ for all $\vec{x}$, $\vec{y} \in \mathbb{R}^m$.

   Hint: Diagonalise A. Given $\vec{x}$, can you find $\vec{v}_x$ such that $\|\vec{x}\|_A = \|\vec{v}_x\|$? Is the mapping $\vec{x} \to \vec{v}_x$ linear?
4. (7pts) Prove that a sequence $\{\vec{x}_n\}$ of points in $\mathbb{R}^m$ converges to a point $\vec{p}$ with respect to the A-norm if and only if it converges with respect to the usual Euclidean norm.

Question 4

i) (10pts) State, with all necessary hypotheses, the implicit function theorem.

ii) (8pts) Show that

$$x^2 + 7xy + 10y^2 - 1 = 0$$

defines y as an continuously differentiable function of x near $(1, 0)$.

iii) (8pts) Are there any points around which neither x or y can be expressed as a continuous function of the other?

iv) (4pts) Determine the maximal set containing $(1, 0)$ on which y can be expressed as a continuously differentiable function of x.