MAU22203-1

Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
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Faculty of Science, Technology, Engineering and Mathematics

School of Mathematics

SF Mathematics **Michaelmas Term 2024**
SF Joint Honours
JS TP

Analysis in Several Real Variables - Sample Exam 1 - Solutions

Day **Place** **Time**

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Instructions to candidates:

Attempt any three questions. If you attempt all four questions, only your best three will be considered in your grade. All questions are worth 30 points

Unless stated otherwise, you may use all statements given lectures without proof, but must clearly justify that the assumptions of statement are fulfilled.

Additional instructions for this examination:

You may use a non-programmable calculator. Please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

Question 1

For a linear operator $T : \mathbb{R}^m \to \mathbb{R}^m$, define the operator norm by

$$\|T\|_{op} := \sup_{\|\vec{x}\|=1} \{\|T\vec{x}\|\}.$$

i) (6pts) Explain why the operator norm is well defined, and show there exists $\vec{x} \in \mathbb{R}^m$ such that $\|\vec{x}\| = 1$ and $\|T\vec{x}\| = \|T\|_{op}$.

ii) (10pts) Show that, for any $\vec{x} \in \mathbb{R}^m$ and any linear operators A, $B : \mathbb{R}^m \to \mathbb{R}^m$

$$\|AB\vec{x}\| \leq \|A\|_{op}\|B\|_{op}\|\vec{x}\| \quad \text{and} \quad \|(A+B)\vec{x}\| \leq (\|A\|_{op} + \|B\|_{op})\,\|\vec{x}\|$$

and hence that

$$\|AB\|_{op} \leq \|A\|_{op}\|B\|_{op} \quad \text{and} \quad \|A+B\|_{op} \leq \|A\|_{op} + \|B\|_{op}.$$

iii) (10pts) Let $T : \mathbb{R}^m \to \mathbb{R}^m$ be a linear transformation with norm $\|T\|_{op} = \lambda < 1$. Show that for any $\varepsilon > 0$, there exists $N > 0$ such that, for all $n > m \geq N$,

$$\Big\| \sum_{k=m+1}^{n} T^k \Big\|_{op} < \varepsilon$$

iv) (4pts) Hence show that the series

$$\lim_{n\to\infty} \sum_{k=1}^{n} T^k \vec{x}$$

converges for every $\vec{x}$

Hint: Recall that $\left\{x_n = \sum_{k=1}^{n} x^k\right\}$ *is a convergent sequence for all* $x \in [0, 1)$.

Solution

i) Linear operators are continuous, as is taking norms. Hence the map

$$\vec{x} \mapsto \|T\vec{x}\|$$

is a continuous map. The set $\{\vec{x} \in \mathbb{R}^m \mid \|\vec{x}\| = 1\}$ is a closed set, as it is the inverse image of the closed set $\{1\}$ via the norm map, and it is clearly bounded. Hence, by the Extreme Value Theorem

$$\sup_{\|\vec{x}\|=1} \{\|T\vec{x}\|\}$$

is well defined, and there exists $\vec{x}$ of norm 1 achieving this supremum.

ii) As shown in lectures

$$\|T\vec{x}\| \leq \|T\|_{op}\|\vec{x}\|$$

for any $\vec{x} \in \mathbb{R}^m$.

Hence

$$\|AB\vec{x}\| \leq \|A\|_{op}\|B\vec{x}\| \leq \|A\|_{op}\|B\|_{op}\|\vec{x}\|.$$

Thus, $\|A\|_{op}\|B\|_{op}$ is an upper bound for the value of $\|AB\vec{x}\|$ on the set of unit vectors. As $\|AB\|_{op}$ is the least upper bound on this set, we must have

$$\|AB\|_{op} \leq \|A\|_{op}\|B\|_{op}.$$

By the triangle inequality

$$\|(A+B)\vec{x}\| = \|A\vec{x} + B\vec{x}\| \leq \|A\vec{x}\| + \|B\vec{x}\| \leq \|A\|_{op}\|\vec{x}\| + \|B\|_{op}\|\vec{x}\|.$$

Similarly to the case of multiplication, this implies that $\|A\|_{op} + \|B\|_{op}$ is an upper bound for the value of $\|(A+B)\vec{x}\|$ on the set of unit vectors. As $\|A+B\|_{op}$ is the least upper bound on this set, we must have

$$\|A+B\|_{op} \leq \|A\|_{op} + \|B\|_{op}.$$

iii) By (ii), we can bound this norm by

$$\|\sum_{k=m+1}^{n} T^k\|_{op} \leq \sum_{k=m+1}^{n} \|T\|_{op}^k = \sum_{k=m+1}^{n} \lambda^k.$$

Since $\lambda \in [0,1)$, the sequence $\{\sum_{k=1}^{n} \lambda^k\}$ converges and is therefore a Cauchy sequence. Hence, for all $\varepsilon > 0$, there exists $N > 0$ such that, for all $n > m \geq N$

$$\sum_{k=m+1}^{n} \lambda^k = |\sum_{k=1}^{n} \lambda^k - \sum_{k=1}^{m} \lambda^k| < \varepsilon$$

and hence, for all $n > m \geq N$

$$\|\sum_{k=m+1}^{n} T^k\|_{op} < \varepsilon$$

iv) Part (iii) implies that the sequence

$$\left\{\vec{x}_n := \sum_{k=1}^{n} T^k \vec{x}\right\}$$

is a Cauchy sequence. If $\vec{x} = \vec{0}$, then this is the constant sequence. Otherwise, for $n > m$,

$$\|\vec{x}_n - \vec{x}_m\| \leq \Big\| \sum_{k=m+1}^{n} T^k \Big\|_{op} \|\vec{x}\}$$

which can be made arbitrarily small: given $\varepsilon > 0$, choose $N > 0$ such that the operator norm is less than $\frac{\varepsilon}{\|\vec{x}\|}$ for all $n > m \geq N$. As every Cauchy sequence converges, $\vec{x}_n$ converges.

Question 2

i) (8pts) State, with all necessary hypotheses, Rolle's theorem.

ii) (12pts) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$. Show that if $f'(a) \neq 0$ for any $a \in \mathbb{R}$, then f is injective.

iii) (10pts) Define the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x, y) = \begin{pmatrix} e^x \cos y \\ e^x \sin y \end{pmatrix}.$$

Determine the Jacobian matrix $(\mathrm{J}\, F)_{(x,y)}$ and show that this is invertible at every point of $\mathbb{R}^2$. Is F injective?

Solution

i) Let $f : [a, b] \to \mathbb{R}$ be a function that is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b) = 0$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

ii) We will prove the contrapositive. Suppose f is not injective. Then there exists $a, b \in \mathbb{R}$ such that $f(a) = f(b) = C$ for some $C \in \mathbb{R}$. Define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x) - C$. As f is differentiable everywhere, it is continuous everywhere. Hence g is continuous on $[a, b]$ and differentiable on (a, b). Furthermore, $g(a) = g(b) = 0$. Thus there exists $c \in (a, b)$ such that $g'(c) = 0$. But then

$$f'(c) = g'(c) = 0.$$

Hence, the contrapositive holds, and so the original statement holds.

iii) The Jacobian matrix is

$$\begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

which has determinant $e^x(\cos^2 y + \sin^2 y) = e^x$, which is strictly positive for all $x \in \mathbb{R}$. Thus, the Jacobian has non-zero determinant at every point of $\mathbb{R}^2$ and is therefore invertible.

However, F is not injective, as $F(x, y + 2\pi) = F(x, y)$ for all $(x, y) \in \mathbb{R}^2$.

Question 3

i) (8pts) Let $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ be a function and let $\vec{p}$ be a point of $\mathbb{R}^m$. Define what it means for φ to be differentiable with derivative $T : \mathbb{R}^m \to \mathbb{R}^n$ at $\vec{p}$.

ii) (6pts) Prove that a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at every point $\vec{p} \in \mathbb{R}^m$ and determine its derivative.

iii) (8pts) We call $f : \mathbb{R}^2 \to \mathbb{R}^n$ bilinear if

$$f(ax_1 + x_2, by_1 + y_2) = abf(x_1, y_1) + af(x_1, y_2) + bf(x_2, y_1) + f(x_2, y_2)$$

for all $a, b, x_1, x_2, y_1, y_2 \in \mathbb{R}$. Prove that

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k)|}{\|(h,k)\|} = 0.$$

iv) (8pts) Hence prove that the derivative of a bilinear map $f : \mathbb{R}^2 \to \mathbb{R}$ at a point $(p, q) \in \mathbb{R}^2$ is the linear transformation

$$(\mathrm{D}\, f)_{(p,q)}(x, y) = f(p, y) + f(x, q).$$

Solution

i) The function φ is differentiable at $\vec{p}$ with derivative T if and only if

$$\lim_{\vec{x}\to\vec{p}} \frac{\varphi(\vec{x}) - \varphi(\vec{p}) - T(\vec{x} - \vec{p})}{\|\vec{x} - \vec{p}\|} = 0.$$

ii) Note that, by linearity, $T\vec{x} - T\vec{p} - T(\vec{x} - \vec{p}) = 0$ for all $\vec{x}, \vec{p} \in \mathbb{R}^m$. Hence

$$\lim_{\vec{x}\to\vec{p}} \frac{T\vec{x} - T\vec{p} - T(\vec{x} - \vec{p})}{\|\vec{x} - \vec{p}\|} = 0.$$

As such, T is differentiable at every point $\vec{p}$ with derivative T.

iii) By bilinearity $f(h, k) = hkf(1, 1)$. Thus

$$\frac{|f(h,k)|}{\|(h,k)\|} = \frac{|hk||f(1,1)|}{\sqrt{h^2 + k^2}}$$

and so it suffices to show that

$$\lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2+k^2}} = 0$$

Note that, since $(|h|-|k|)^2 \geq 0$, $|hk| \leq \frac{1}{2}(h^2+k^2)$ and so

$$0 \leq \lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2+k^2}} \leq \lim_{(h,k)\to(0,0)} \frac{h^2+k^2}{2\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{1}{2}\sqrt{h^2+k^2} = 0$$

and so

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k)|}{\|(h,k)\|} = 0.$$

iv) It suffices to show that

$$\lim_{(x,y)\to(p,q)} \frac{f(x,y) - f(p,q) - (\mathrm{D}\, f)_{(p,q)}\left((x-p, y-q)\right)}{\|(x-p, y-q)\|} = 0,$$

or equivalently, that

$$\lim_{(h,k)\to(0,0)} \frac{f(p+h, q+k) - f(p,q) - (\mathrm{D}\, f)_{(p,q)}(h,k)}{\|(h,k)\|} = 0.$$

Expanding the numerator via bilinearly and the definition of the given linear transformation, we see that it is equal to

$$f(p,q) + f(p,k) + f(h,q) + f(h,k) - f(p,q) - f(p,k) - f(h,q) = f(h,k).$$

Then part (iii) tells us that

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)}{\|(h,k)\|} = 0.$$

Thus f is differentiable with the given derivative.

Question 4

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$, and let $F : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function on $\mathbb{R}^2$.

i) (8pts) Show that there exists $c \in (0, 1)$ such that

$$\int_0^1 f(x)dx = f(c).$$

ii) (12pts) Hence prove that there exists $(c, d) \in (0, 1)^2$ such that

$$\int_0^1 \int_0^1 F(x, y)dx\, dy = F(c, d)$$

Hint: try to do this in two steps, but be careful to keep track of what depends on x or y and what is actually constant!

iii) (10pts) Determine such a point $(c, d) \in (0, 1)^2$ for $F(x, y) = x^2 + 2xy + 3y^2$.

Solution

i) By the fundamental theorem of calculus, the function

$$t \mapsto \int_0^t f(x)dx$$

is a continuous function on $[0, 1]$ and is differentiable on $(0, 1)$, with derivative $f(x)$. Hence, by the mean value theorem, there exists $c \in (0, 1)$ such that

$$\frac{\int_0^1 f(x)dx - \int_0^0 f(x)dx}{1 - 0} = f(c)$$

The left hand side is just $\int_0^1 f(x)dx$, as $\int_0^0 f(x)dx = 0$.

ii) Fix $y \in [0, 1]$. Then the function $F(x, y)$ is a continuous function of x, and hence, by part (i), there exists $c(y) \in (0, 1)$ such that

$$\int_0^1 F(x, y)dx = F(c_y, y)$$

From the lectures, we know that

$$\int_0^1 F(x, y)dx$$

is a continuous function of y, and hence $F(c(y), y)$ is a continuous function of y. Therefore, by part (i), there exists $d \in (0,1)$ such that

$$\int_0^1 \int_0^1 F(x,y)dx\,dy = \int_0^1 F(c(y),y)dy = F(c(d),d)$$

Letting c be the constant $c(d) \in (0,1)$, we find

$$\int_0^1 \int_0^1 F(x,y)dx\,dy = F(c,d).$$

iii) Let us first find $c(y)$ such that

$$\int_0^1 F(x,y)dx = F(c(y),y).$$

We can easily compute

$$\int_0^1 (x^2 + 2xy + 3y^2)\,dx = \frac{1}{3} + y + 3y^2$$

Hence, we want to determine $c(y) \in (0,1)$ such that

$$c(y)^2 + 2c(y)y + 3y^2 = \frac{1}{3} + y + 3y^2$$

or equivalently such that

$$c(y)^2 + 2c(y)y - \frac{1}{3} - y = 0$$

Viewing this as a quadratic equation, we find

$$c(y) = -y \pm \sqrt{y^2 + y + \frac{1}{3}}$$

which is only positive if we take the positive square root.

We then want to find d such that

$$\int_0^1 \int_0^1 F(x,y)dx\,dy = F(c(d),d).$$

The integral evaluates to

$$\int_0^1 \int_0^1 F(x,y)dx\,dy = \frac{1}{3} + \frac{1}{2} + 1 = \frac{11}{6}$$

while

$$F(c(d),d) = \int_0^1 F(x,d)dx = c(d)^2 + 2c(d)d + 3d^2 = \frac{1}{3} + d + 3d^2$$

from the definition of $c(d)$. Hence, we must have

$$3d^2 + d + \frac{1}{3} = \frac{11}{6}$$

and hence

$$d = \frac{-2 \pm \sqrt{4 + 72}}{12} = \frac{-1 \pm \sqrt{19}}{6}$$

and we must take the positive square root to ensure $d \in (0, 1)$. It is easy to check that $\frac{-1+\sqrt{19}}{6} \in (0, 1)$ as it is positive, and $\sqrt{19} - 1 < 4 < 6$.

Thus, $c = c(d) = -d + \sqrt{d^2 + d + \frac{1}{3}}$, which we can compute to be awful, but $d^2 + d + \frac{1}{3} < d^2 + 2d + 1$, so $c(d) \in (0, 1)$. Precisely

$$c = \frac{-1 + \sqrt{19}}{6} + \sqrt{\frac{13 + 2\sqrt{19}}{18}}$$