Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Science, Technology, Engineering and Mathematics

School of Mathematics

SF Mathematics **Michaelmas Term 2024**
SF Joint Honours
JS TP

Analysis in Several Real Variables - Sample Exam 1

Day **Place** **Time**

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Instructions to candidates:

Attempt any three questions. If you attempt all four questions, only your best three will be considered in your grade. All questions are worth 30 points

Unless stated otherwise, you may use all statements given lectures without proof, but must clearly justify that the assumptions of statement are fulfilled.

Additional instructions for this examination:

You may use a non-programmable calculator. Please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

Question 1

For a linear operator $T : \mathbb{R}^m \to \mathbb{R}^m$, define the operator norm by

$$\|T\|_{op} := \sup_{\|\vec{x}\|=1} \{\|T\vec{x}\|\}.$$

i) (6pts) Explain why the operator norm is well defined, and show there exists $\vec{x} \in \mathbb{R}^m$ such that $\|\vec{x}\| = 1$ and $\|T\vec{x}\| = \|T\|_{op}$.

ii) (10pts) Show that, for any $\vec{x} \in \mathbb{R}^m$ and any linear operators $A,\, B : \mathbb{R}^m \to \mathbb{R}^m$

$$\|AB\vec{x}\| \leq \|A\|_{op}\|B\|_{op}\|\vec{x}\| \quad \text{and} \quad \|(A+B)\vec{x}\| \leq \left(\|A\|_{op} + \|B\|_{op}\right)\|\vec{x}\|$$

and hence that

$$\|AB\|_{op} \leq \|A\|_{op}\|B\|_{op} \quad \text{and} \quad \|A+B\|_{op} \leq \|A\|_{op} + \|B\|_{op}.$$

iii) (10pts) Let $T : \mathbb{R}^m \to \mathbb{R}^m$ be a linear transformation with norm $\|T\|_{op} = \lambda < 1$. Show that for any $\varepsilon > 0$, there exists $N > 0$ such that

$$\|\sum_{k=m+1}^{n} T^k\| < \varepsilon$$

iv) (4pts) Hence show that the series

$$\lim_{n\to\infty} \sum_{k=1}^{n} T^k$$

converges.

Hint: Recall that $\left\{x_n = \sum_{k=1}^{n} x^k\right\}$ *is a convergent sequence for all* $x \in [0, 1)$.

Question 2

i) (8pts) State, with all necessary hypotheses, Rolle's theorem.

ii) (12pts) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$. Show that if $f'(a) \neq 0$ for any $a \in \mathbb{R}$, then f is injective.

iii) (10pts) Define the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x, y) = \begin{pmatrix} e^x \cos y \\ e^x \sin y \end{pmatrix}.$$

Determine the Jacobian matrix $(\mathrm{J}\, F)_{(x,y)}$ and show that this is invertible at every point of $\mathbb{R}^2$. Is F injective?

Question 3

i) (8pts) Let $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ be a function and let $\vec{p}$ be a point of $\mathbb{R}^m$. Define what it means for φ to be differentiable with derivative $T : \mathbb{R}^m \to \mathbb{R}^n$ at $\vec{p}$.

ii) (6pts) Prove that a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at every point $\vec{p} \in \mathbb{R}^m$ and determine its derivative.

iii) (8pts) We call $f : \mathbb{R}^2 \to \mathbb{R}^n$ bilinear if

$$f(ax_1 + x_2, by_1 + y_2) = abf(x_1, y_1) + af(x_1, y_2) + bf(x_2, y_1) + f(x_2, y_2)$$

for all $a, b, x_1, x_2, y_1, y_2 \in \mathbb{R}$. Prove that

$$\lim_{(h,k)\to(0,0)} \frac{|f(h, k)|}{\|(h, k)\|} = 0.$$

iv) (8pts) Hence prove that the derivative of a bilinear map $f : \mathbb{R}^2 \to \mathbb{R}$ at a point $(p, q) \in \mathbb{R}^2$ is the linear transformation

$$(\mathrm{D}\, f)_{(p,q)}(x, y) = f(p, y) + f(x, q).$$

Question 4

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$, and let $F : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function on $\mathbb{R}^2$.

i) (8pts) Show that there exists $c \in (0, 1)$ such that

$$\int_0^1 f(x)dx = f(c).$$

ii) (12pts) Hence prove that there exists $(c, d) \in (0, 1)^2$ such that

$$\int_0^1 \int_0^1 F(x, y)dxdy = F(c, d)$$

Hint: try to do this in two steps, but be careful to keep track of what depends on x *or* y *and what is actually constant!*

iii) (10pts) Determine such a point $(c, d) \in (0, 1)^2$ for $F(x, y) = x^2 + 2xy + 3y^2$.