MAU22203/33203 - Analysis in Several Real Variables

Tutorial Sheet 3

Trinity College Dublin

Course homepage

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Matrix norms and exponentials*

i) Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map. Show that the map $\vec{x} \to T\vec{x}$ is continuous at every point of $\mathbb{R}^m$.

   Hint: Recall the second question of the homework. Consider the zero map as a separate case

ii) Recall the Hilbert-Schmidt norm of a real $(m \times n)$-matrix $A = (A_{i,j})$ is defined by

$$\|A\|_{HS} = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{i,j}^2}.$$

   Show that the Hilbert-Schmidt norm satisfies the triangle inequality:

$$\|A + B\|_{HS} \leq \|A\|_{HS} + \|B\|_{HS}.$$

   Hint: Either replicate the proof for the Euclidean norm, or figure out a way to view this as a Euclidean norm

iii) Let T be an $(n \times n)$-matrix. Define a sequence of matrices by $\{T_s = \sum_{k=0}^{s} \frac{1}{k!} T^k\}$. Viewing this as a sequence in $\mathbb{R}^{n^2}$, show that this is a Cauchy sequence and hence converges. You may freely use the fact that

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges for all real $\mathbb{R}$.

iv) Define $\exp(T) : \lim_{s \to \infty} T_s$ as the limit of this sequence. Can you find an example of two square matrices A, B such that

$$\exp(A + B) \neq \exp(A) \exp(B)?$$

Solution 1

i) We will show that this map is continuous at an arbitrary point $\vec{p} \in \mathbb{R}^m$. Let $\varepsilon > 0$ be given. If T is the zero map, then

$$\|T\vec{x} - T\vec{p}\| = 0 < \varepsilon$$

for every $\vec{x} \in \mathbb{R}^m$, and in particular for every point such that $\|\vec{x} - \vec{p}\| < 1$.

If T is not the zero map, then recall that $\|T\vec{v}\| \leq \|T\|_{HS} \|\vec{v}\|$ for every $\vec{v} \in \mathbb{R}^m$, and also note that, since the Hilbert Schmidt norm is defined in terms of the sums of the squares of entries of T, $\|T\|_{HS} \neq 0$ for any non-zero T. As such, for every $\vec{x} \in \mathbb{R}^m$ such that

$$\|\vec{x} - \vec{p}\| < \frac{\varepsilon}{\|T\|_{HS}}$$

we have that

$$\|T\vec{x} - T\vec{p}\| = \|T(\vec{x} - \vec{p})\| \leq \|T\|_{HS} \|\vec{x} - \vec{p}\| < \|T\|_{HS} \frac{\varepsilon}{\|T\|_{HS}} = \varepsilon.$$

ii) The vector space of $(m \times n)$-matrices is clearly isomorphic to $\mathbb{R}^{mn}$ under the map taking a matrix to the vector obtained by stacking the columns of the matrix to one long vector. Furthermore, the Hilbert-Schmidt norm of a matrix is automatically equal to the Euclidean norm of the corresponding vector, and hence the triangle inequality follows from the triangle inequality for $\mathbb{R}^{mn}$.

iii) If we can show that this is a Cauchy sequence in $\mathbb{R}^{n^2}$, then we know all such sequences converge. To see that it is Cauchy, note that, for $t > s$,

$$\begin{aligned}\|T_t - T_s\| &= \|\sum_{k=s+1}^{t} \frac{1}{k!} T^k\| \leq \sum_{k=s+1}^{t} \frac{1}{k!} \|T^k\| \\ &= \sum_{k=s+1}^{t} \frac{1}{k!} \|T^k\|_{HS} \leq \sum_{k=s+1}^{t} \frac{1}{k!} \|T\|_{HS}^k\end{aligned}$$

We know that the series $\sum_{k=0}^{\infty} \frac{1}{k!} \|T\|_{HS}^k$ converges, which is to say that the sequence

$$\{x_s = \sum_{k=0}^{s} \frac{1}{k!} \|T\|_{HS}^k\}$$

converges. This implies that it is Cauchy, so for every $\varepsilon > 0$, there exists $N > 0$ such that for all $t > s \geq N$,

$$\|T_t - T_s\| \leq \sum_{k=s+1}^{t} \frac{1}{k!} \|T\|_{HS}^k = |x_t - x_s| < \varepsilon$$

where we have used the non-negativity of $\|T\|_{HS}$. Hence the sequence $\{T_s\}$ is Cauchy and convergent.

iv) We will take

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

so that

$$\exp(A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$\exp(B) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Hence

$$\exp(A)\exp(B) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

while

$$\exp(A+B) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \cdots = \begin{pmatrix} \cosh(1) & \sinh(1) \\ \sinh(1) & \cosh(1) \end{pmatrix}$$

Exercise 2 *Mean value theorem*

i) Does the mean value theorem apply to the function $f(x) = 4\sqrt{x}$ on $[0, 1]$? If so, determine a value of c satisfying the conclusions of the theorem.

ii) A man is driving in a 80km/h zone. He enters the zone at 13:00, and has travelled 405km in a single direction by 18:00. Did he obey the speed limit at all times?

iii) Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Show that these exists $c \in (a, b)$ such that

$$\int_a^b f(x)dx = f(c)(b - a).$$

Solution 2

i) f is continuous on $[0, 1]$, and differentiable with derivative $f' \frac{2}{\sqrt{x}}$ on $(0, 1)$, so there exists $c \in (0, 1)$ such that

$$f'(c) = \frac{4 - 0}{1 - 0} = 4$$

Then note that $f'(\frac{1}{4}) = 4$, so we can take $c = \frac{1}{4}$.

ii) We assume that this man is driving in a continuous and differentiable fashion, so that his velocity is the derivative of his displacement. As he is driving in one direction, his displacement equals his distance. Define $s(t)$ to be the distance travelled after t hours, so that his velocity is $s'(t)$. By the mean value theorem, there exists a number of hours c such that

$$s'(c) = \frac{405 - 0}{18 - 13} = 81\text{km/h}$$

and thus the man must have been briefly speeding.

iii) The function $F(x) = \int_a^x f(y)dy$ defined by the integral is continuous on $[a, b]$ and differentiable on (a, b) with derivative $f(x)$. Furthermore $F(a) = 0$. Therefore, by the mean value theorem, there exists $c \in (a, b)$ such that

$$f(c) = \frac{F(b) - F(a)}{b - a} = \frac{\int_a^b f(x)dx}{b - a}$$

from which the claim follows.

Exercise 3 *Differentiability and first order approximation*

i) Compute the partial derivatives of $f(x,y) = \frac{x^3y}{1+x^2+y^4}$.

ii) Compute the partial derivatives of $f(x,y) = x^y$ at a point where $x > 0$ and $y > 0$.

iii) Compute the partial derivatives of $f(x,y) = \sin(x\sin(y))$.

iv) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that f is differentiable at x_0 if and only if there exist a, $b \in \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{f(x_0+h) - a - bh}{h} = 0.$$

Solution 3

i) We will just provide the answers here.

$$\frac{\partial f}{\partial x} = \frac{x^4y + 3x^2y + 3x^2y^5}{(1+x^2+y^4)^2}, \quad \frac{\partial f}{\partial y} = \frac{x^3 + x^5 - 3x^3y^4}{(1+x^2+y^4)^2}.$$

ii)

$$\frac{\partial f}{\partial x} = yx^{y-1}, \quad \frac{\partial f}{\partial y} = x^y \ln x$$

iii)

$$\frac{\partial f}{\partial x} = \sin(y)\cos(x\sin(y)), \quad \frac{\partial f}{\partial y} = x\cos(x\sin(y))\cos(y).$$

iv) Suppose f is differentiable at x_0, and let $a = f(x_0)$ and $b = f'(x_0)$. Then

$$\begin{aligned}\lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - f'(x_0)h}{h} &= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} - f'(x_0) \\ &= f'(x_0) - f'(x_0) = 0,\end{aligned}$$

where we used the existence of both parts of the limit to split it into two.

Now, suppose this limit condition holds. The only way this limit can exist is if

$$\lim_{h\to 0} f(x_0+h) - a - bh = 0$$

which, by continuity of f, implies that $a = f(x_0)$. Then, the limit condition implies that, for every $\varepsilon > 0$ there exists δ such that

$$\left| \frac{f(x_0 + h) - f(x_0)}{h} - b \right| < \varepsilon$$

for all $0 < |h| < \delta$. This is precisely what it means for

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

to exist and be equal to b, i.e. f is differentiable at x_0 with derivative b.

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