

# MAU22203/33203 - Analysis in Several Real Variables

## Tutorial Sheet 1

Trinity College Dublin

Course homepage

The use of electronic calculators and computer algebra software is allowed.

### **Exercise 1** *Convergence in $\mathbb{R}$*

- (i) What is the least upper bound of the set  $\{x \in \mathbb{R} \mid x^2 - 3x + 2 < 0\}$ ?
- (ii) What is the greatest lower bound of the set  $\{\sin(x) + \cos(x) \mid 0 \leq x \leq \pi\}$ ?
- (iii) Define a sequence  $\{x_n = \sum_{k=0}^n \frac{1}{k!}\}$ . Is this monotonic? Bounded? Convergent?
- (iv) Determine the limit (or argue that no such limit exists) of

$$\left\{ x_n = \frac{2^n - 2^{-n}}{3^n + 3^{-n}} \right\}.$$

### **Solution 1**

- (i) The quadratic equation  $x^2 - 3x + 2$  has roots  $x = 1$  and  $x = 2$ , and is easily seen to be positive for  $x \notin [1, 2]$ . Therefore the set

$$\{x \mid x^2 - 3x + 2 < 0\} = (1, 2)$$

which has least upper bound 2.

(ii) The function  $f(x) = \sin(x) + \cos(x)$  is continuous and defined on a closed interval  $[0, \pi]$ , and therefore achieves its minimum on this interval. At  $x = 0$  or  $x = \pi$ ,  $f(x) = \pm 1$ , so it remains to check for any internal minima.  $f$  has turning points wherever  $f'(x) = \cos(x) - \sin(x) = 0$ . In the interval  $(0, \pi)$ , this holds only for  $x = \frac{\pi}{4}$  and  $x = \frac{3\pi}{4}$ . Checking these, we see that  $f(\frac{3\pi}{4}) = -\sqrt{2} < -1$ , so the greatest lower bound is  $-\sqrt{2}$ .

(iii) The sequence  $\{x_n\}$  is monotonically increasing, as  $x_{n+1} = x_n + \frac{1}{(n+1)!}$  is obtained by adding a positive term. It is bounded above, since

$$n! \geq 2^{n-1} \Leftrightarrow \frac{1}{n!} \leq 2^{1-n}$$

for all  $n \geq 1$  and so

$$x_n \leq \sum_{k=0}^n 2^{1-k} < 4$$

and bounded below by 0. Hence it is convergent (to  $e$ , but that's a different problem).

(iv) Note that the sequence is bounded below by 0, and that since

$$\begin{aligned} 2^n - 2^{-n} &< 2^n, \\ 3^n + 3^{-n} &> 3^n, \end{aligned}$$

we must have that  $x_n < \left(\frac{2}{3}\right)^n$ , which tends to 0 as  $n \rightarrow \infty$ . Hence, we must have that  $\lim_{n \rightarrow \infty} x_n = 0$ .

## Exercise 2 *Practice with norms*

(i) Show that, for  $\vec{x}, \vec{y} \in \mathbb{R}^m$ ,

$$2\|\vec{x}\|^2 + 2\|\vec{y}\|^2 = \|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2.$$

(ii) Given  $\vec{x}_1 \neq \vec{x}_2 \in \mathbb{R}^m$  and  $0 < c < 1$ , prove that there exists  $\vec{y} \in \mathbb{R}^m$  and  $r \in \mathbb{R}$  such that the sets

$$\{\vec{x} \in \mathbb{R}^m \mid \|\vec{x} - \vec{x}_1\| = c\|\vec{x} - \vec{x}_2\|\}$$

and

$$\{\vec{x} \in \mathbb{R}^m \mid \|\vec{x} - \vec{y}\| = r\}$$

are equal.

## Solution 2

(i) Recall that  $\|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle$ . Thus

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2 &= \langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle + \langle \vec{x} - \vec{y}, \vec{x} - \vec{y} \rangle \\ &= \langle \vec{x}, \vec{x} \rangle + 2\langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{y} \rangle \\ &\quad + \langle \vec{x}, \vec{x} \rangle - 2\langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{y} \rangle \\ &= 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2. \end{aligned}$$

(ii) Squaring the defining condition of the first set (which changes nothing, as all quantities are non-negative), gives us that

$$\langle \vec{x} - \vec{x}_1, \vec{x} - \vec{x}_1 \rangle = c^2 \langle \vec{x} - \vec{x}_2, \vec{x} - \vec{x}_2 \rangle.$$

Expanding this out using bilinearity, and rearranging terms, we get

$$(1 - c^2)\|\vec{x}\|^2 - 2\langle \vec{x}_1 - c^2\vec{x}_2, \vec{x} \rangle + \|\vec{x}_1\|^2 - c^2\|\vec{x}_2\|^2 = 0$$

Defining  $\vec{y} := \frac{1}{1-c^2}\vec{x}_1 - \frac{c^2}{1-c^2}\vec{x}_2$ , we note that

$$\begin{aligned} \|\vec{x} - \vec{y}\|^2 &= \|\vec{x}\|^2 - 2\langle \vec{y}, \vec{x} \rangle + \|\vec{y}\|^2 \\ &= \|\vec{y}\|^2 + \frac{c}{1-c^2}\|\vec{x}_2\|^2 - \frac{1}{1-c^2}\|\vec{x}_1\|^2 \end{aligned}$$

as required, taking  $r^2$  to be the right hand side. In order for  $r$  to be a real number, we need to check that the right hand side is non-negative, but this follows quickly by expanding out  $\|\vec{y}\|^2$  to obtain that

$$r^2 = \frac{c^2}{(1-c^2)^2}\|\vec{x}_2\|^2 - \frac{2c^2}{(1-c^2)^2}\langle \vec{x}_2, \vec{x}_1 \rangle + \frac{c^2}{(1-c^2)^2} = \frac{c^2\|\vec{x}_2 - \vec{x}_1\|^2}{(1-c^2)^2} \geq 0$$

## Exercise 3 *Fun with Cauchy-Schwarz*

1. Show that for any  $x_1, \dots, x_m \in \mathbb{R}$ ,

$$(x_1 + x_2 + \dots + x_m)^2 \leq m(x_1^2 + \dots + x_m^2)$$

2. Show that, for any  $x, y \in \mathbb{R}$

$$(x + y)^2 \leq (x^2 + 1)(y^2 + 1).$$

### Solution 3

- (i) This is precisely the CS inequality for  $\vec{x} = (x_1, \dots, x_m)$  and  $\vec{y} = (1, \dots, 1)$ .
- (ii) This is precisely the CS inequality for  $\vec{x} = (x, 1)$ ,  $\vec{y} = (1, y)$ .

### Exercise 4 Sequences in $\mathbb{R}^m$

Determine the limit, or argue that it does not exist, of the following sequences

- (i)  $\{(x_{n,1}, x_{n,2}) = (\frac{1+n}{1-2n}, \frac{1}{n^3})\}$
- (ii)  $\{(x_{n,1}, x_{n,2}) = (1 - 2^{-n}, n \sin(n^{-1}))\}$
- (iii)  $\{(x_{n,1}, x_{n,2}) = ((1 - n^{-1}) \cos(\frac{2\pi n}{7}), (1 + n^{-1}) \sin(\frac{2\pi n}{7}))\}$

*In the above questions, you may compute one variable limits using any technique you like. A formal  $\varepsilon - N$  proof is not required.*

### Solution 4

- (i) In the first coordinate, we can divide above and below by  $n$  to get that

$$\lim_{n \rightarrow \infty} \frac{1+n}{1-2n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} + 1}{\frac{1}{n} - 2} = -\frac{1}{2}.$$

The second coordinate is strictly decreasing and must tend to 0. Hence  $\vec{x}_n \rightarrow (-\frac{1}{2}, 0)$ .

- (ii) As  $2^{-n} \rightarrow 0$ , the first coordinate tends to 1. As  $\frac{\sin(x)}{x} \rightarrow 1$  as  $x \rightarrow 0$ , we must have that

$$n \sin(n^{-1}) = \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \rightarrow 1$$

as  $n$  tends to  $\infty$ . Hence  $\vec{x}_n \rightarrow (1, 1)$ .

- (iii) Note that, while  $1 - n^{-1} \rightarrow 1$ ,  $\cos(\frac{2\pi n}{7})$  cycles through 13 distinct values, and therefore the first component, and hence the sequence  $\{\vec{x}_n\}$ , cannot converge.