MAU22203/33203 - Analysis in Several Real Variables

Tutorial Sheet 3

Trinity College Dublin

Course homepage

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Matrix norms and exponentials*

i) Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map. Show that the map $\vec{x} \to T\vec{x}$ is continuous at every point of $\mathbb{R}^m$.

   Hint: Recall the second question of the homework. Consider the zero map as a separate case

ii) Recall the Hilbert-Schmidt norm of a real $(m \times n)$-matrix $A = (A_{i,j})$ is defined by

$$\|A\|_{HS} = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{i,j}^2}.$$

   Show that the Hilbert-Schmidt norm satisfies the triangle inequality:

$$\|A + B\|_{HS} \leq \|A\|_{HS} + \|B\|_{HS}.$$

   Hint: Either replicate the proof for the Euclidean norm, or figure out a way to view this as a Euclidean norm

iii) Let T be an $(n \times n)$-matrix. Define a sequence of matrices by $\{T_s = \sum_{k=0}^{s} \frac{1}{k!} T^k\}$. Viewing this as a sequence in $\mathbb{R}^{n^2}$, show that this is a Cauchy sequence and hence converges. You may freely use the fact that

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges for all real $\mathbb{R}$.

iv) Define $\exp(T) : \lim_{s \to \infty} T_s$ as the limit of this sequence. Can you find an example of two square matrices A, B such that

$$\exp(A + B) \neq \exp(A) \exp(B)?$$

Exercise 2 *Mean value theorem*

i) Does the mean value theorem apply to the function $f(x) = 4\sqrt{x}$ on $[0, 1]$? If so, determine a value of c satisfying the conclusions of the theorem.

ii) A man is driving in a 80km/h zone. He enters the zone at 13:00, and has travelled 405km in a single direction by 18:00. Did he obey the speed limit at all times?

iii) Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Show that these exists $c \in (a, b)$ such that

$$\int_a^b f(x)dx = f(c)(b - a).$$

Exercise 3 *Differentiability and first order approximation*

i) Compute the partial derivatives of $f(x, y) = \frac{x^3 y}{1+x^2+y^4}$.

ii) Compute the partial derivatives of $f(x, y) = x^y$ at a point where $x > 0$ and $y > 0$.

iii) Compute the partial derivatives of $f(x, y) = \sin(x \sin(y))$.

iv) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that f is differentiable at x_0 if and only if there exist a, $b \in \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{f(x_0 + h) - a - bh}{h} = 0.$$