

MAU22203/33203 - Analysis in Several Real Variables

Tutorial Sheet 2

Trinity College Dublin

Course homepage

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Convergent subsequences*

Let $\{\vec{x}_k\}$ be a bounded sequence in \mathbb{R}^m that does not converge to $\vec{0}$.

- i) Show that there exist $R, c > 0$ such that $\{\vec{x}_k\}$ contains a subsequence contained within the set

$$\{\vec{x} \in \mathbb{R}^m \mid c \leq \|\vec{x}\| \leq R\},$$

- ii) Show that this subsequence contains a convergent subsequence, converging to a point not equal to $\vec{0}$,
- iii) Hence conclude that $\{\vec{x}_k\}$ contains a convergent subsequences, converging to a point other than $\vec{0}$.

Exercise 2 *Interiors*

Given a subset $X \subset \mathbb{R}^m$, we define its interior X° , closure \overline{X} , and boundary ∂X by

$$\begin{aligned} X^\circ &= \bigcup_{\substack{U \subset X \\ U \text{ open}}} U, \\ \overline{X} &= \bigcap_{\substack{F \supset X \\ F \text{ closed}}} F, \\ \partial X &= \overline{X} \setminus X^\circ. \end{aligned}$$

- i) Show that X° is open and \overline{X} is closed,
- ii) Find the interior, closure, and boundary of

$$\{\vec{x} \in \mathbb{R}^m \mid \|\vec{x}\| \leq 2\},$$

- iii) Find the interior, closure, and boundary of

$$\{(x, y) \in \mathbb{R}^2 \mid x \geq 2, y > 0\},$$

- iv) Find the interior, closure, and boundary of

$$\{x \in \mathbb{R} \mid x \in \mathbb{Q}\}.$$

Hint: It might be easier to consider the interior of the complement.

Exercise 3 *Topological stuff*

- i) Identify whether the following sets are open/closed in \mathbb{R}^2 :
 - a) $\{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$,
 - b) $\{(x, y) \in \mathbb{R}^2 \mid x \neq 0, y \neq 0\}$,
 - c) $\{(x, y) \in \mathbb{R}^2 \mid \max\{|x|, |y|\} = 1\}$,
 - d) $\{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 17xy < 34\}$

- ii) Suppose that $X \subset \mathbb{R}^m$ is not closed. Show that there exists a point $\vec{v} \in \mathbb{R}^m \setminus X$ such that

$$B(\vec{v}, \varepsilon) \cap X \neq \emptyset$$

for all $\varepsilon > 0$.

- iii) Call a subset A of \mathbb{Z} open if $\mathbb{Z} \setminus A$ is finite or if A is empty. Show that this collection of open sets defines a topology on \mathbb{Z} .

Exercise 4 *Continuous functions and convergent sequences*

- i) Without resorting to an $\varepsilon - \delta$ proof, show that if $\phi_1, \phi_2 : \mathbb{R}^m \rightarrow \mathbb{R}^n$ are continuous functions, then

$$\vec{x} \mapsto \phi_1(\vec{x}) + \phi_2(\vec{x})$$

is a continuous function.

- ii) Without resorting to an $\varepsilon - N$ argument, prove that if we have two sequences $\{\vec{x}_k\}$ and $\{\vec{y}_k\}$ in \mathbb{R}^m converging to \vec{p} and \vec{q} respectively, then the sequence in \mathbb{R} given by

$$\{z_k = \langle \vec{x}_k, \vec{y}_k \rangle\}$$

converges to $\langle \vec{p}, \vec{q} \rangle$.

- iii) Let $X \subset \mathbb{R}^m$ be a strict subset of \mathbb{R}^m , and suppose we have a point $\vec{p} \in \mathbb{R}^m \setminus X$. Show that

$$\phi_{\vec{p}}(\vec{x}) = \frac{1}{\|\vec{x} - \vec{p}\|}$$

is continuous on X .

- iv) Hence argue that if X is not closed in \mathbb{R}^m , there exists $\vec{p} \in \mathbb{R}^m \setminus X$ such that the image $\phi_{\vec{p}}(X)$ is not bounded in \mathbb{R} .

Hint: Use 3.ii to construct a bad sequence.