MAU22203/33203 - Analysis in Several Real Variables

Tutorial Sheet 1

Trinity College Dublin

Course homepage

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Convergence in $\mathbb{R}$*

(i) What is the least upper bound of the set $\{x \in \mathbb{R} \mid x^2 - 3x + 2 < 0\}$?

(ii) What is the greatest lower bound of the set $\{\sin(x) + \cos(x) \mid 0 \leq x \leq \pi\}$?

(iii) Define a sequence $\{x_n = \sum_{k=0}^{n} \frac{1}{k!}\}$. Is this monotonic? Bounded? Convergent?

(iv) Determine the limit (or argue that no such limit exists) of

$$\left\{ x_n = \frac{2^n - 2^{-n}}{3^n + 3^{-n}} \right\}.$$

Exercise 2 *Practice with norms*

(i) Show that, for $\vec{x}$, $\vec{y} \in \mathbb{R}^m$,

$$2\|\vec{x}\|^2 + 2\|\vec{y}\|^2 = \|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2.$$

(ii) Given $\vec{x}_1 \neq \vec{x}_2 \in \mathbb{R}^m$ and $0 < c < 1$, prove that there exists $\vec{y} \in \mathbb{R}^m$ and $r \in \mathbb{R}$ such that the sets

$$\{\vec{x} \in \mathbb{R}^m \mid \|\vec{x} - \vec{x}_1\| = c\|\vec{x} - \vec{x}_2\|\}$$

and

$$\{\vec{x} \in \mathbb{R}^m \mid \|\vec{x} - \vec{y}\| = r\}$$

are equal.

Exercise 3 *Fun with Cauchy-Schwarz*

1. Show that for any $x_1, \ldots, x_m \in \mathbb{R}$,

$$(x_1 + x_2 + \cdots + x_m)^2 \leq m\left(x_1^2 + \cdots + x_m^2\right)$$

2. Show that, for any $x, y \in \mathbb{R}$

$$(x + y)^2 \leq (x^2 + 1)(y^2 + 1).$$

Exercise 4 *Sequences in* $\mathbb{R}^m$

Determine the limit, or argue that it does not exist, of the following sequences

(i) $\{(x_{n,1}, x_{n,2}) = (\frac{1+n}{1-2n}, \frac{1}{n^3})\}$

(ii) $\{(x_{n,1}, x_{n,2}) = (1 - 2^{-n}, n\sin(n^{-1}))\}$

(iii) $\{(x_{n,1}, x_{n,2}) = \left((1 - n^{-1})\cos(\frac{2\pi n}{7}), (1 + n^{-1})\sin(\frac{2\pi n}{7})\right)\}$

In the above questions, you may compute one variable limits using any technique you like. A formal $\varepsilon - N$ *proof is not required.*