MAU22203/33203 - Analysis in Several Real Variables

Exercise Sheet 4

Trinity College Dublin

Course homepage

This is an entirely optional homework. If submitted, the best 3 out of 4 homeworks will be considered for your continuous assessment. Answers are due for December 1^{st}, 23:59.

Exercise 1 *An extended fundamental theorem of calculus (60pts)*

i) (20pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a Riemann integrable function, and let $b : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Determine the derivative of

$$g(x) = \int_0^{b(x)} f(t)\, dt.$$

Hint: Recall that the function

$$x \mapsto \int_a^x f(t)\, dt$$

is differentiable with derivative $f(x)$. Chain rule?

ii) (10pts) Let $a : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Determine the derivative of

$$h(x) = \int_{a(x)}^0 f(t)\, dt.$$

iii) (20pts) Hence determine the derivative of

$$r(x) = \int_{a(x)}^{b(x)} f(t)\,dt.$$

iv) (10 pts) Hence, or otherwise, compute the derivative of

$$q(x) = \int_{1-x^2}^{1+x^2} \sin(t^3 - 3t^2 + 3t)\,dt$$

Solution 1

i) Denote by $F : \mathbb{R} \to \mathbb{R}$ the function

$$F(x) = \int_0^x f(t)\,dt$$

which is differentiable with derivative $f(x)$. Note that $g(x) = F(b(x))$, so by chain rule

$$g'(x) = (F \circ b)'(x) = F'(b(x))b'(x) = f(b(x))b'(x).$$

ii) Note that

$$h(x) = \int_{a(x)}^0 f(t)\,dt = -\int_0^{a(x)} f(t)\,dt$$

and so, by the previous part

$$h'(x) = -f(a(x))a'(x).$$

iii) We have that

$$r(x) = \int_0^{b(x)} f(t)\,dt + \int_{a(x)}^0 f(t)\,dt = g(x) + h(x)$$

and so

$$r'(x) = f(b(x))b'(x) - f(a(x))a'(x).$$

iv) We compute that

$$q'(x) = \sin((1+x^2)^3 - 3(1+x^2)^2 + 3(1+x^2))(2x) - \sin((1-x^2)^3 - 3(1-x^2)^2 + 3(1-x^2))(-2x)$$

which simplifies to

$$q'(x) = 2x\left(\sin(1+x^6) + \sin(1-x^6)\right) = 4\sin(1)\cos(x^6)$$

An essentially identical calculation shows that

$$\int_{\alpha-x^2}^{\alpha+x^2} \sin(t^3 - 3\alpha t^2 + 3\alpha^2 t)\, dt$$

is constant for $\alpha = \sqrt[3]{\pi}$. Wild, right?

Exercise 2 *Applying the implicit function theorem (40pts)*

1. (10pts) Let $f : [0,1]^3 \to \mathbb{R}$ be a continuous function. Using Fubini's theorem for 2 variable functions, show that

$$\int_0^1 \int_0^1 \int_0^1 f(x,y,z)\, dx\, dy\, dz = \int_0^1 \int_0^1 \int_0^1 f(x,y,z)\, dy\, dz\, dx$$

Hint: Remember the integrand must be continuous to use Fubini

2. (10 pts) Show that

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)^2}{x^2+y^2} \text{ if } (x,y) \neq (0,0), \\ 0 \text{ otherwise} \end{cases}$$

is continuous on $[0,1]^2$.

Hint: Bound the absolute value near 0 in terms of norms.

3. (20 pts) Compute

$$\int_{[0,1]^2} \frac{xy(x^2-y^2)^2}{x^2+y^2}\, dA$$

Hint: You may (and should) use without proof that

$$\int_0^1 t^5\left(\ln(t^2+1) - \ln(t^2)\right)\, dt = \frac{\ln(2)}{3} - \frac{1}{12}$$

Solution 2

i) For fixed z, the function $f(x, y, z)$ is continuous as a function of x and y. Hence

$$\int_0^1 \int_0^1 f(x, y, z)\, dx\, dy = \int_0^1 \int_0^1 f(x, y, z)\, dy\, dx$$

and so

$$\begin{aligned}
\int_0^1 \int_0^1 \int_0^1 f(x, y, z)\, dx\, dy\, dz &= \int_0^1 \left(\int_0^1 \int_0^1 f(x, y, z)\, dx\, dy \right) dz \\
&= \int_0^1 \left(\int_0^1 \int_0^1 f(x, y, z)\, dy\, dx \right) dz \\
&= \int_0^1 \int_0^1 \int_0^1 f(x, y, z)\, dy\, dx\, dz.
\end{aligned}$$

As $\int_0^1 f(x, y, z)\, dy$ is a continuous function of x and z, we have that

$$\begin{aligned}
\int_0^1 \int_0^1 \int_0^1 f(x, y, z)\, dy\, dx\, dz &= \int_0^1 \left(\int_0^1 \int_0^1 f(x, y, z)\, dy \right) dx\, dz \\
&= \int_0^1 \left(\int_0^1 \int_0^1 f(x, y, z)\, dy \right) dz, dx \\
&= \int_0^1 \int_0^1 \int_0^1 f(x, y, z)\, dy\, dz\, dx.
\end{aligned}$$

ii) The function is clearly continuous away from $(0, 0)$ so it suffices to show that

$$\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0) = 0.$$

Note that, for $(x, y) \neq (0, 0)$

$$\begin{aligned}
|f(x, y)| &= |xy| \frac{|x^2 - y^2|^2}{|x^2 + y^2|} \\
&\leq |xy| \frac{(|x|^2 + |y|^2)^2}{|x^2 + y^2|} \\
&= |xy| frac(x^2 + y^2)^2 x^2 + y^2 \\
&= |xy|(x^2 + y^2) = |xy| \cdot \|(x, y)\|^2
\end{aligned}$$

which clearly tends to 0 as $(x, y) \to (0, 0)$. Thus,

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0$$

and f is continuous.

iii) Since f is continuous, we can compute this as the iterated integral

$$L = \int_0^1 \int_0^1 \frac{xy(x^2 - y^2)^2}{x^2 + y^2}\, dx\, dy.$$

Letting $u = x^2 + y^2$ in the x-integral, we find that

$$\begin{aligned}
L &= \frac{1}{2} \int_0^1 \int_{y^2}^{y^2+1} \frac{y(u - 2y^2)^2}{u}\, du\, dy \\
&= \frac{1}{2} \int_0^1 \int_{y^2}^{y^2+1} yu - 4y^3 + \frac{4y^5}{u}\, du\, dy \\
&= \frac{1}{4} \int_0^1 y - 6y^3 + 8y^5 \ln(y^2 + 1) - 8y^5 \ln(y^2)\, dy \\
&= \frac{1}{4} \left(\frac{1}{2} - \frac{3}{2} + \frac{8 \ln(2)}{3} - \frac{2}{3} \right) \\
&= \frac{2 \ln(2)}{3} - \frac{5}{12}
\end{aligned}$$