MAU22203/33203 - Analysis in Several Real Variables

Exercise Sheet 3

Trinity College Dublin

Course homepage

Answers are due for November 24^{th}, 23:59.
The use of electronic calculators and computer algebra software is allowed.
If typesetting your work, the following code can be used to product a 3×4 matrix, and can be easily modified to produce other sizes of matrices:

```
\[ A= \begin{pmatrix}
    a & b & c & d \\
    e & f & g & h \\
    i & j & k & l
    \end{pmatrix}
\]
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Exercise 1 *Applications of the inverse function theorem (70pts)*

i) (15pts) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map

$$f(x, y) = \begin{pmatrix} y \sin(x^2) \\ e^{xy} \end{pmatrix}$$

Determine the derivative $(\mathrm{D}\, f)_{(x,y)}$ at all points at which f is differentiable.

ii) (15pts) We say that f is locally invertible at $(x, y) \in \mathbb{R}^2$ if there exists open $U \ni (x, y)$ such that $f : U \to f(U)$ has a continuously differentiable inverse. Let A be the set of points (x, y) at which f is locally invertible. Is A open, closed, or either?

Hint: Can we write A as the inverse image of a set under a continuous map?

iii) (10pts) Let $(x_0, y_0) \in A$. What is the derivative of the local inverse of f at $f(x_0, y_0)$?

iv) (30 pts) Let $\phi : X \to \mathbb{R}^m$ be a continuously differentiable map on an open set X with $\det(\mathrm{D}\,\phi)_{\vec{x}} \neq 0$ for every $\vec{x} \in X$. Show that the image $\phi(X)$ is open in $\mathbb{R}^m$

Hint: What does the inverse function theorem say other than the existence of an inverse? Must there be an open ball around every point of the image?

Solution 1

i) We first compute the Jacobian

$$(\mathrm{J}\,f)_{(x,y)} = \begin{pmatrix} 2xy\cos(x^2) & \sin(x^2) \\ ye^{xy} & xe^{xy} \end{pmatrix}$$

As all entries of the Jacobian are continuous on $\mathbb{R}^2$, f is differentiable everywhere in $\mathbb{R}^2$ with derivative $(\mathrm{D}\,f)_{(x,y)} = (\mathrm{J}\,f)_{(x,y)}$.

ii) Since f is continuously differentiable, the inverse function theorem tells us that f is locally invertible when it has invertible derivative. The derivative is invertible when it has non-zero determinant, and so

$$A = \{(x, y) \mid \det(\mathrm{D}\,f)_{(x,y)} \neq 0\}.$$

If the determinant is a continuous function of (x, y), then this set is open, as it is the inverse image of the open set $\mathbb{R} \setminus \{0\}$. We can check that

$$\det(\mathrm{D}\,f)_{(x,y)} = e^{xy}\left(2x^2y\cos(x^2) - y\sin(x^2)\right)$$

is definitely continuous, so A is open.

iii) The derivative of the local inverse is given by the inverse of the derivative, and hence is equal to

$$\frac{1}{e^{xy}\left(2x^2 y\cos(x^2) - y\sin(x^2)\right)}\begin{pmatrix} xe^{xy} & -\sin(x^2) \\ -ye^{xy} & 2xy\cos(x^2) \end{pmatrix}$$

iv) From the inverse function theorem, there exists open $Y \subset \phi(X)$ around every $\vec{q} = \phi(\vec{p})$ in $\phi(X)$. As Y is open, there exists $\varepsilon > 0$ such that

$$B(\vec{q}, \varepsilon) \subset Y \subset \phi(X)$$

for every point $\vec{q} \in \phi(X)$. Hence $\phi(X)$ is open in $\mathbb{R}^m$.

Exercise 2 *Applying the implicit function theorem (30pts)*

1. (20pts) Let
$$S = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^3 + x^2\}$$
Let S_0 be the set of points $(x_0, y_0) \in S$ for which the assumptions of the implicit function theorem apply. Determine S_0. Is S_0 open or closed (or neither) in S?

2. (10 pts) Explain why, for open $V = B((0,0), \frac{1}{2})$, there does not exist an open interval (a, b) such that $V \cap S$ is homeomorphic to (a, b). You do not have to give a rigorous proof.

   Hint: Look at the graph and count the number of points in the boundaries.

Solution 2

i) The function $f(x, y) = y^2 - x^3 - x^2$ is continuously differentiable with derivative
$$(\mathrm{J}\, f)_{(x_0,y_0)} = \begin{pmatrix}-3x^2 - 2x, 2y\end{pmatrix}$$
which has rank 1 unless
$$3x^2 + 2x = 2y = 0 \text{ which implies that} x \in \{0, \frac{-2}{3}\},\ y = 0$$
Since $f(\frac{-2}{3}, 0) \neq 0$, $(0,0)$ is the only point in S such that $(\mathrm{J}\, f)_{(x_0,y_0)}$ has rank 0. Thus, $S_0 = S \setminus \{(0,0)\}$. Furthermore, since $U = \mathbb{R}^2 \setminus \{(0,0)\}$ is open in $\mathbb{R}^2$, $S_0 = S \cap U$ is open in S.

ii) Looking at the graph of of $f(x, y) = 0$, we see that $V \cap S$ is roughly a cross, which clearly has 4 boundary points, which an interval in $\mathbb{R}$ has only 2 boundary points. A homeomorphism would preserve the number of boundary points as it preserves all topological information, so no such homeomorphism can exist.