MAU22203/33203 - Analysis in Several Real Variables

Exercise Sheet 1

Trinity College Dublin

Course homepage

Answers are due for November 3rd, 23:59.
The use of electronic calculators and computer algebra software is allowed, though reasonably thorough computations are expected in Exercise 1, i.e. present the differentiation, though you can simplify using a computer.

Exercise 1 *Order of second order partial derivatives (60pts)*

Given a function $f : \mathbb{R}^m \to \mathbb{R}$ with first order partial derivatives, we can attempt to define second order partial derivatives by

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}) = (\partial_i \partial_j f)(\vec{x}) := \lim_{h \to 0} \frac{(\partial_j f)(\vec{x} + h\vec{e_i}) - (\partial_j f)(\vec{x})}{h}$$

whenever this limit exists. In more standard notation

$$\frac{\partial^2 f}{\partial x_i \partial x_j} := \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial j}\right).$$

When the first order partial derivatives are continuous, if the second order partial derivatives, when they are defined, are also continuous, the order in which we take the derivatives does not matter. In general

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \neq \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) := \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} \text{ if } (x,y) \neq (0,0), \\ 0 \text{ if } (x,y) = (0,0). \end{cases}$$

i) (30pts) Compute the partial derivatives of f with respect to x and y away from $(0,0)$, using the standard rules, and at $(0,0)$, using the limit definition.

ii) (30 pts) Compute $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ away from $(0,0)$ and at $(0,0)$, using the limit definition if necessary.

Solution 1

i) Away from $(0,0)$, the usual rules of differentiation apply, so we can apply quotient/product rule to determine that

$$\frac{\partial f}{\partial x}(x,y) = \frac{3x^2y - y^3}{x^2+y^2} - 2x\left(\frac{x^3y - xy^3}{(x^2+y^2)^2}\right),$$

which simplifies to

$$\frac{\partial f}{\partial x}(x,y) = \frac{3x^4y + 3x^2y^3 - x^2y^3 - y^5 - 2x^4y + 2x^2y^3}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}.$$

At $(0,0)$, we need to compute the limit directly

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{0-0}{h} = 0. \end{aligned}$$

For the partial derivative with respect to y, we could compute this directly, or note that that

$$f(x,y) = -f(y,x)$$

and apply chain rule to get that, away from $(0,0)$,

$$\frac{\partial f}{\partial y}(x,y) = -\frac{\partial f}{\partial x}(y,x) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}.$$

At $(0,0)$, we can again apply this symmetry argument or compute the limit directly:

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0}\frac{0-0}{h} = 0.\end{aligned}$$

ii) Away from $(0,0)$, we can apply the usual rules:

$$\begin{aligned}\frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial}{\partial x}\left(\frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}\right)\\ &= \frac{5x^4-12x^2y^2-y^4}{(x^2+y^2)^2} - 4x\left(\frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^3}\right)\\ &= \frac{5x^6+5x^4y^2-12x^4y^2-12x^2y^4-x^2y^4-y^6-4x^6+16x^4y^2+4x^2y^4}{(x^2+y^2)^3}\\ &= \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}.\end{aligned}$$

Noting that

$$\frac{\partial f}{\partial x}(x,y) = -\frac{\partial f}{\partial y}(y,x)$$

we have that

$$\frac{\partial^2 f}{\partial y\partial x} = -\frac{\partial^2 f}{\partial x\partial y}(y,x) = \frac{\partial^2 f}{\partial x\partial y}(x,y)$$

since

$$\frac{\partial^2 f}{\partial x\partial y}(x,y) = -\frac{\partial^2 f}{\partial x\partial y}(y,x).$$

To compute the derivatives at $(0,0)$, we compute the limit directly

$$\begin{aligned}\frac{\partial^2 f}{\partial x\partial y}(0,0) &= \lim_{h\to 0}\frac{\frac{h^5}{h}-0}{h} = 1,\\ \frac{\partial^2 f}{\partial y\partial x}(0,0) &= \lim_{h\to 0}\frac{\frac{-h^5}{h}-0}{h} = -1,\end{aligned}$$

Exercise 2 *A special case of Lagrange multipliers (40pts)*

1. (30pts) Let
$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$
be the unit circle, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function with first order partial derivatives. Suppose that f, restricted to a function on the circle, has a local extremum at a point $(x_0, y_0) \notin \{(\pm 1, 0), (0, \pm 1)\}$. By writing
$$S^1 = \{(\cos(t), \sin(t)) \mid t \in [0, 2\pi]\}$$
show that there exists $\lambda \in \mathbb{R}$ such that
$$\frac{\partial f}{\partial x}(x_0, y_0) = 2x_0\lambda, \quad \frac{\partial f}{\partial y}(x_0, y_0) = 2y_0\lambda.$$
Hint: Since $x_0, y_0 \neq 0$, we can define λ satisfying one of these equations. Consider the function $F(t) = f(\cos(t), \sin(t))$. Freely using chain rule for such compositions, what does the derivative of F look like at a local extremum? Does that help us?

2. (10 pts) Using this determine the possible extrema of $f(x, y) = x^3 + y^3$ restricted to $x^2 + y^2 = 1$, and the associated λ.

   Remark: This means without using the parameterisation of S^1 in terms of trigonometric functions!

Solution 2

i) The function f restricted to S^1 can be viewed as a function on $[0, 2\pi]$ by composing f with $(\cos(t), \sin(t))$. Define
$$F(t) := f(\cos(t), \sin(t)).$$
Choose $t_0 \in (0, 2\pi)$ such that $(\cos(t_0), \sin(t_0)) = (x_0, y_0)$, we have that F has a local extremum at t_0. From the extreme value theorem, this implies that $F'(t_0) = 0$, as F is differentiable on $(0, 2\pi)$. By the chain rule, this implies that
$$F'(t_0) = -\frac{\partial f}{\partial x}(x_0, y_0)\cos(t_0) + \frac{\partial f}{\partial y}(x_0, y_0)\sin(t_0) = 0$$

which is to say that

$$\frac{\partial f}{\partial y}(x_0, y_0)y_0 = \frac{\partial f}{\partial x}(x_0, y_0)x_0.$$

Since $y_0 \neq 0$, we can define $\lambda = \frac{1}{2y_0}\frac{\partial f}{\partial y}(x_0, y_0)$, so that λ satisfies the second equation. We can then check that

$$\begin{aligned} 2x_0\lambda &= \frac{x_0}{y_0}\frac{\partial f}{\partial y}(x_0, y_0) \\ &= \frac{1}{y_0}\left(y_0\frac{\partial f}{\partial x}(x_0, y_0)\right) = \frac{\partial f}{\partial x}(x_0, y_0). \end{aligned}$$

ii) From the previous question, any local extremum occurs at $(\pm 1, 0)$, $(0, \pm 1)$ or a point (x_0, y_0) such that

$$x_0^2 + y_0^2 = 1$$

and there exists λ

$$\frac{\partial f}{\partial x}(x_0, y_0) = 2x_0\lambda, \quad \frac{\partial f}{\partial y}(x_0, y_0) = 2y_0\lambda.$$

Thus, we want to solve the system

$$3x^2 = 2x\lambda, \quad 3y^2 = 2y\lambda, \quad x^2 + y^2 = 1.$$

For $x, y \neq 0$, we can solve this for x and y in terms of λ, which we can plug into the circle equation to determine λ. This gives us possible extrema at

$$\begin{aligned} (x, y, \lambda) &= \pm(1, 0, \frac{3}{2}), \\ (x, y, \lambda) &= \pm(0, 1, \frac{3}{2}), \\ (x, y, \lambda) &= \pm(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{3}{2\sqrt{2}}) \end{aligned}$$

all of which give $x^3 + y^3 = \pm 1$ depending on the sign of λ.