MAU22203/33203 - Analysis in Several Real Variables

Exercise Sheet 4

Trinity College Dublin

Course homepage

This is an entirely optional homework. If submitted, the best 3 out of 4 homeworks will be considered for your continuous assessment. Answers are due for December 1st, 23:59.

Exercise 1 *An extended fundamental theorem of calculus (60pts)*

i) (30pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a Riemann integrable function, and let $b : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Determine the derivative of

$$g(x) = \int_0^{b(x)} f(t)\,dt.$$

Hint: Recall that the function

$$x \mapsto \int_a^x f(t)\,dt$$

is differentiable with derivative $f(x)$. Chain rule?

ii) (10pts) Let $a : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Determine the derivative of

$$h(x) = \int_{a(x)}^0 f(t)\,dt.$$

iii) (20pts) Hence determine the derivative of

$$r(x) = \int_{a(x)}^{b(x)} f(t)\,dt.$$

iv) (40 pts) Hence, or otherwise, compute the derivative of

$$q(x) = \int_{1-x^2}^{1+x^2} \sin(t^3 - 3t^2 + 3t)\,dt$$

Exercise 2 *Applying the implicit function theorem (40pts)*

1. (10pts) Let $f : [0,1]^3 \to \mathbb{R}$ be a continuous function. Using Fubini's theorem for 2 variable functions, show that

$$\int_0^1 \int_0^1 \int_0^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_0^1 \int_0^1 f(x,y,z)\,dy\,dz\,dx$$

   Hint: Remember the integrand must be continuous to use Fubini

2. (10 pts) Show that

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)^2}{x^2+y^2} \text{ if } (x,y) \neq (0,0), \\ 0 \text{ otherwise} \end{cases}$$

   is continuous on $[0,1]^2$.

   Hint: Bound the absolute value near 0 in terms of norms.

3. (20 pts) Compute

$$\int_{[0,1]^2} \frac{xy(x^2-y^2)^2}{x^2+y^2}\,dA$$

   Hint: You may (and should) use without proof that

$$\int_0^1 t^5 \left(\ln(t^2+1) - \ln(t^2)\right)\,dt = \frac{\ln(2)}{3} - \frac{1}{12}$$