MAU22203/33203 - Analysis in Several Real Variables

Exercise Sheet 3

Trinity College Dublin

Course homepage

Answers are due for November 24th, 23:59.
The use of electronic calculators and computer algebra software is allowed.
If typesetting your work, the following code can be used to product a 3×4 matrix, and can be easily modified to produce other sizes of matrices:

```
\[ A= \begin{pmatrix}
    a & b & c & d \\
    e & f & g & h \\
    i & j & k & l
    \end{pmatrix}
\]
```

Exercise 1 *Applications of the inverse function theorem (70pts)*

i) (15pts) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map

$$f(x,y) = \begin{pmatrix} y\sin(x^2) \\ e^{xy} \end{pmatrix}$$

Determine the derivative $(\mathrm{D}\, f)_{(x,y)}$ at all points at which f is differentiable.

ii) (15pts) We say that f is locally invertible at $(x,y) \in \mathbb{R}^2$ if there exists open $U \ni (x,y)$ such that $f : U \to f(U)$ has a continuously differentiable inverse. Let A be the set of points (x, y) at which f is locally invertible. Is A open, closed, or either?

Hint: Can we write A as the inverse image of a set under a continuous map?

iii) (10pts) Let $(x_0, y_0) \in A$. What is the derivative of the local inverse of f at $f(x_0, y_0)$?

iv) (30 pts) Let $\phi : X \to \mathbb{R}^m$ be a continuously differentiable map on an open set X with $\det(\mathrm{D}\,\phi)_{\vec{x}} \neq 0$ for every $\vec{x} \in X$. Show that the image $\phi(X)$ is open in $\mathbb{R}^m$

Hint: What does the inverse function theorem say other than the existence of an inverse? Must there be an open ball around every point of the image?

Exercise 2 *Applying the implicit function theorem (30pts)*

1. (20pts) Let
$$S = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^3 + x^2\}$$
Let S_0 be the set of points $(x_0, y_0) \in S$ for which the assumptions of the implicit function theorem apply. Determine S_0. Is S_0 open or closed (or neither) in S?

2. (10 pts) Explain why, for open $V = B((0, 0), \frac{1}{2})$, there does not exist an open interval (a, b) such that $V \cap S$ is homeomorphic to (a, b). You do not have to give a rigorous proof.

   Hint: Look at the graph and count the number of points in the boundaries.