MAU22203/33203 - Analysis in Several Real Variables

Exercise Sheet 1

Trinity College Dublin

Course homepage

Answers are due for November 3rd, 23:59.
The use of electronic calculators and computer algebra software is allowed, though reasonably thorough computations are expected in Exercise 1, i.e. present the differentiation, though you can simplify using a computer.

Exercise 1 *Order of second order partial derivatives (60pts)*

Given a function $f : \mathbb{R}^m \to \mathbb{R}$ with first order partial derivatives, we can attempt to define second order partial derivatives by

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}) = (\partial_i \partial_j f)(\vec{x}) := \lim_{h \to 0} \frac{(\partial_j f)(\vec{x} + h\vec{e_i}) - (\partial_j f)(\vec{x})}{h}$$

whenever this limit exists. In more standard notation

$$\frac{\partial^2 f}{\partial x_i \partial x_j} := \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial j}\right).$$

When the first order partial derivatives are continuous, if the second order partial derivatives, when they are defined, are also continuous, the order in which we take the derivatives does not matter. In general

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \neq \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) := \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} \text{ if } (x,y) \neq (0,0), \\ 0 \text{ if } (x,y) = (0,0). \end{cases}$$

i) (30pts) Compute the partial derivatives of f with respect to x and y away from $(0,0)$, using the standard rules, and at $(0,0)$, using the limit definition.

ii) (30 pts) Compute $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ away from $(0,0)$ and at $(0,0)$, using the limit definition if necessary.

Exercise 2 *A special case of Lagrange multipliers (40pts)*

1. (30pts) Let

$$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

be the unit circle, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function with first order partial derivatives. Suppose that f, restricted to a function on the circle, has a local extremum at a point $(x_0, y_0) \notin \{(\pm 1, 0), (0, \pm 1)\}$. By writing

$$S^1 = \{(\cos(t), \sin(t)) \mid t \in [0, 2\pi]\}$$

show that there exists $\lambda \in \mathbb{R}$ such that

$$\frac{\partial f}{\partial x}(x_0, y_0) = 2x_0\lambda, \quad \frac{\partial f}{\partial y}(x_0, y_0) = 2y_0\lambda.$$

Hint: Since $x_0, y_0 \neq 0$, we can define λ satisfying one of these equations. Consider the function $F(t) = f(\cos(t), \sin(t))$. Freely using chain rule for such compositions, what does the derivative of F look like at a local extremum? Does that help us?

2. (10 pts) Using this determine the possible extrema of $f(x,y) = x^3 + y^3$ restricted to $x^2 + y^2 = 1$, and the associated λ.

Remark: This means without using the parameterisation of S^1 in terms of trigonometric functions!