MAU22203/33203 - Analysis in Several Real Variables

Exercise Sheet 1

Trinity College Dublin

Course homepage

Answers are due for October 4$^{\text{th}}$, 2pm.
The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Properties of sequences (40pts)*

Let $\{\vec{x}_n\}$ and $\{\vec{y}_n\}$ be two sequences of points in $\mathbb{R}^m$, and let $\lambda \in \mathbb{R}$ be a real number. Suppose that $\{\vec{x}_n\}$ converges to a point $\vec{p}$, and $\{\vec{y}_n\}$ converges to a point $\vec{q}$. By giving a formal ε-N proof, establish the following:

(10pts) The sequence $\{\lambda\vec{x}_n\}$ converges to $\lambda\vec{p}$,

Hint: consider $\lambda = 0$ as a separate case.

(10pts) The sequence $\{\vec{z}_n = \vec{x}_n + \vec{y}_n\}$ converges to $\vec{p} + \vec{q}$.

(20pts) The sequence of real numbers $\{a_n = \langle \vec{x}_n, \vec{y}_n \rangle\}$ given by the inner product of $\vec{x}_n$ with $\vec{y}_n$ converges to the inner product $\langle \vec{p}, \vec{q} \rangle$.

Hint: Using the following, apply the Cauchy-Schwarz inequality

$$\langle \vec{x}, \vec{y} \rangle - \langle \vec{p}, \vec{q} \rangle = \langle \vec{x} - \vec{p}, \vec{y} - \vec{q} \rangle + \langle \vec{p}, \vec{y} - \vec{q} \rangle + \langle \vec{x} - \vec{p}, \vec{q} \rangle.$$

Exercise 2 *Matrix norms (60pts)*

In the following, you may use any standard facts from your first year courses. Let $A : \mathbb{R}^m \to \mathbb{R}^s$ be a linear transformation represented by the $(s \times m)$-matrix $(A_{i,j})_{\substack{1\le i\le s\\ 1\le j\le m}}$ with respect to the standard bases. We define the Hilbert-Schmidt norm of T by

$$\|A\|_{HS} := \sqrt{\sum_{i=1}^{s}\sum_{j=1}^{m} A_{i,j}^2}.$$

(15 pts) Show that, for any $\vec{x} \in \mathbb{R}^m$,

$$\|A\vec{x}\| \le \|A\|_{HS}\|\vec{x}\|.$$

Hint: what are the components of $A\vec{x}$ and how could we bound them using the Cauchy-Schwarz inequality?

(15 pts) Show that, given linear transformations

$$A : \mathbb{R}^m \to \mathbb{R}^s \quad \text{and} B : \mathbb{R}^s \to \mathbb{R}^t,$$

the Hilbert Schmidt norm satisfies

$$\|BA\|_{HS} \le \|B\|_{HS}\|A\|_{HS}$$

(15 pts) Denoting by A^T the transpose of the matrix A, and by $\mathrm{tr}(M)$ the trace of a square matrix M, show that

$$\|A\|_{HS}^2 = \mathrm{tr}(A^T A)$$

(15 pts) Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the below matrix, and define a sequence of points in $\mathbb{R}^3$ by $\vec{x}_n := A^{n-1}\vec{x}_1$, where $\vec{x}_1$ is given below. Prove that $\{\vec{x}_n\}$ converges to $\vec{0}$.

$$A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{10} & 0 \end{pmatrix}, \quad \vec{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$