Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Science, Technology, Engineering and Mathematics

School of Mathematics

SF/JS Mathematics **Michaelmas Term 2024**

Introduction to Number Theory - Sample Exam 1

Day **Place** **Time**

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Instructions to candidates:

Attempt any three questions. If you attempt all four questions, only your best three will be considered in your grade. All questions are worth 30 points

Unless stated otherwise, you may use all statements given lectures without proof, but must clearly justify that the assumptions of statement are fulfilled.

Additional instructions for this examination:

Formula and tables are available from the inviligators if required.

You may use a non-programmable calculator. Please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

Question 1

i) (15pts) Show that for p a prime number

$$(p-1)! \equiv -1 \pmod{p}.$$

Hint: Try to pair $1, 2, \ldots, p-1$ *up with their multiplicative inverse modulo* p*. Consider* $p = 2$ *separately.*

ii) (7 pts) Show that for n a composite number

$$(n-1)! \not\equiv -1 \pmod{n}.$$

iii) (8pts) Determine $30! \pmod{899}$, noting that $899 = 29 \times 31$.

Question 2

i) (4pts) Factorise $xy + ax + by + ab$.

ii) (8pts) Determine all integers x, $y \in \mathbb{Z}$ such that

$$xy - x - y = 0.$$

iii) (8pts) Determine all integers x, $y \in \mathbb{Z}$ such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{14}.$$

iv) (10pts) Describe all integer solutions x, y, $z \in \mathbb{Z}$ to the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}.$$

Question 3

Let $a, b, c \in \mathbb{Z}$, let $p \in \mathbb{N}$ be an odd prime, and suppose that $p \not| a$.

1. (4pt) State necessary and sufficient conditions for the quadratic equation

$$\overline{a}x^2 + \overline{b}x + \overline{c} = \overline{0}$$

to have 0, 1, and 2 solutions in $\mathbb{Z}/p\mathbb{Z}$ respectively.

2. (7pts) Determine the number of solutions to

$$x^2 - \overline{3}x + \overline{3} = \overline{0}$$

in $\mathbb{Z}/31\mathbb{Z}$.

3. (7pts) Determine the number of solutions to

$$x^2 - \overline{3}x + \overline{3} = \overline{0}$$

in $\mathbb{Z}/37\mathbb{Z}$.

4. (12pts) Hence determine the number of solutions to

$$x^2 - \overline{3}x + \overline{3} = \overline{0}$$

in $\mathbb{Z}/1147\mathbb{Z}$. Be sure to fully justify your answer.

Question 4

i) (12pts) By considering p-adic valuations, show that $\sqrt{2}$ is irrational.

ii) (12pts) By considering p-adic valuations, show that $\log_2 9$ is irrational.

iii) (6pts) Give an example of irrational $x, y \in \mathbb{R}$ such that $x^y \in \mathbb{Q}$ is rational.