MAU22103/33101 - Introduction to Number Theory

Exercise Sheet 4

Trinity College Dublin

Course homepage

Answers are due for Friday November 8^{th}, 2pm.
The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Infinite descent with infinite equations*

Consider the Diophantine equation

$$x^2 + y^2 + z^2 = 2xyz.$$

We want to show that it has no non-negative integer solutions other than $(0, 0, 0)$.

1. (10pts) Show that if (x, y, z) is a solution where at least one of x, y, or z is 0, then they are all 0.

   Knowing this, it suffices to show that we have no positive integer solutions.

2. (10pts) Show that there are no solutions (x, y, z) where exactly 1 or 3 of x,y, and z are odd.

   Hint: Parity

3. (15pts) Show that there are no solutions (x, y, z) where exactly 2 of x, y, and z are odd.

   Hint: Parity²: if one is even, what is $2xyz \pmod 4$*?*

4. (25pts) Show that, given a positive integer solution (x_1, y_1, z_1) to the Diophantine equation given, there exists a positive integer solution to the Diophantine equation

$$x^2 + y^2 + z^2 = 4xyz.$$

5. (25pts) Show that, given a positive integer solution (x_k, y_k, z_k) to the Diophantine equation

$$x^2 + y^2 + z^2 = 2^k xyz$$

   there exists a positive integer solution to the Diophantine equation

$$x^2 + y^2 + z^2 = 2^{k+1} xyz.$$

6. (15pts) Hence conclude that there are no positive integer solutions to

$$x^2 + y^2 + z^2 = 2xyz$$

   Hint: How important is the equation to infinite descent?

This was the only exercise that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them

However, I strongly encourage you to give them a try, as the best way to learn number theory is through practice.

The exercises marked with a star are the exercises I will try to talk about in the tutorial lecture. If there are any exercises would would particularly like to discuss, please let me know

The exercises are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available with the solution to the main exercise.

Exercise 2 *General Pythagorean triples*

Show that every Pythagorean triple (a, b, c) is of the form

$$a = d(u^2 - v^2), \quad b = 2duv, \quad c = d(u^2 + v^2)$$

up to swapping a and b, where $u, v, d \in \mathbb{N}$ satisfy $u > v$, $\gcd(u, v)$ and exactly one of u and v is odd.

Exercise 3 *Hypotenuses of hypotenuses*

i) Show that, to any positive integer solution (a, b, c), with a, b, c pairwise coprime, to

$$x^2 + y^2 = z^4$$

we can associate a primitive Pythagorean triple (s, t, c).

ii) Hence, classify all such solutions to

$$x^2 + y^2 = z^4.$$

Exercise 4 *Odd numbers in Pythagorean triples*

i) Show that, if $t, n \in \mathbb{N}$ satisfy

$$t^2 + (n-1)^2 = n^2$$

then $\gcd(t, n-1) = \gcd(t, n) = 1$. Hence conclude that n is odd.

ii) Writing $n = 2s + 1$, determine for what positive integers s there exists t such that

$$t^2 + (2s)^2 = (2s+1)^2$$

iii) Hence show that every odd number greater than 1 appears in a primitive Pythagorean triple.

Exercise 5 *Areas of Pythagorean triangles*

You may freely use the results of Exercise 2 here.

i) Determine if there exists a right angled triangle with integer side lengths of area 35.

ii) Determine if there exists a right angled triangle with integer side lengths of area 546.

iii) Show that, for $n \geq 54$, there exists a right angled triangle with integer side lengths, and area between n and $2n$.

Hint: Pick a primitive Pythagorean triple (a, b, c) *and consider the right angled triangle with side lengths* (ak, bk, ck) *for* $k \in \mathbb{N}$. *Given* $n \geq 54$, *can we find* k *such that*

$$Area(ak, bk, ck) \leq n < Area(a(k+1), b(k+1), c(k+1))?$$

If so, how can we bound the second area by $2n$*?*

Remark: Integers that are the areas of right angled triangles with rational side lengths are called congruent numbers, and can be classified using special functions called modular forms, which also appear in the proof of Fermat's Last Theorem.

Exercise 6 *Homogeneous equations and rational solutions*

Call a polynomial $F(x_1, x_2, \ldots, x_n)$ homogeneous if and only if there exists $d \geq 0$ such that

$$F(\lambda x_1, \ldots, \lambda x_n) = \lambda^d F(x_1, \ldots, x_n)$$

for all $\lambda \in \mathbb{R}$.

i) Show that $F(x_1, \ldots, x_n)$ has non-zero integer solutions if and only if it has non-zero rational solutions.

ii) Show that $F(x, y, z) = x^2 + y^2 - z^2$ is homogeneous.

iii) As $F(x, y, z)$ is homogeneous, every non-zero integer solution (a, b, c) to $F(x, y, z) = 0$ corresponds uniquely to a point on the circle

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

with rational coordinates As such, it suffices to find all rational points on the circle to describe all Pythagorean triples.

Show that every point other than $(-1, 0)$ on the circle can be written in the form

$$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

for a unique $t \in \mathbb{R}$

iv) Show that a point $(x, y) \neq (-1, 0)$ on the circle has rational coordinates $x, y \in \mathbb{Q}$ if and only if $t \in \mathbb{Q}$.

v) Hence recover our classification of primitive Pythagorean triples.

Hint: Let (a, b, c) be a primitive Pythagorean triple, and write $t = \frac{v}{u}$ where $\gcd u, v = 1$. Why must $u > v > 0$? Then note that if $\frac{a}{c} = \frac{A}{C}$, and $\gcd(a, c) = \gcd(A, C) = 1$, we must have $a = A$ and $c = C$.

Exercise 7 *Infinite descent times three*

Via the method of infinite descent, show that there are no triples of positive integers $a, b, c \in \mathbb{N}$ such that

$$9a^3 + 3b^3 + c^3 = 0$$

Unhelpful Hint: 3

Exercise 8 *Infinite descent for rational solutions*

Consider again the equation

$$x^2 + y^2 + z^2 = 2xyz.$$

We have seen that it has no positive integer solutions. We will now show that it has no non-zero rational solutions

i) Show that there are no non-zero integer solutions, by considering what happening is one, two, or three of x, y, z are negative.

ii) Show that the existence of a rational solution is equivalent to the existence of a triple of rational numbers (x, y, r) such that

$$x^2y^2 - x^2 - y^2 = r^2.$$

iii) Show that the existence of a triple of such rational numbers is equivalent to the existence of a triple (a, b, c) of integers such that

$$a^2b^2 - a^2 - b^2 = c^2$$

Hint: take a common denominator and write $(x, y, r) = (\frac{a}{n}, \frac{b}{n}, \frac{q}{n})$

iv) By rewriting the equation as

$$(a^2 - 1)(b^2 - 1) - 1 = c^2$$

conclude that any integer solution (a, b, c) must have a, b, c all even.

v) Via the same formulation of the equation, show that if (a, b, c) satisfies

$$a^2b^2 - a^2 - b^2 = c^2$$

and $c = 0$, then $a = b = 0$.

Hint: What are the divisors of 1*?*

vi) Argue by infinite descent that no non-zero integer solution to

$$a^2b^2 - a^2 - b^2 = c^2$$

exists.

Hint: Let $a = 2r$*,* $b = 2s$*,* $c = 2t$*. What equation must* (r, s, t) *satisfy? What does that tell us about the parity of* r*,* s*, and* t*?*

vii) Hence conclude that no non-zero rational solution to

$$x^2 + y^2 + z^2 = 2xyz$$

exists.