The block filtration and motivic multiple zeta values

Adam Keilthy
University of Oxford

29/06/2020
Warsaw Number Theory Seminar

Zeta values

Definition

$$\zeta(k) := \sum_{n \geq 1} \frac{1}{n^k},\ k \geq 2$$

Theorem (Euler, 1734)

$\zeta(2k) = (-1)^{k+1} \frac{B_{2k}(2\pi)^{2k}}{2(2k)!}$ *where* B_{2k} *are the Bernoulli numbers.*

Conjecture

The following are algebraically independent:

$$\{\pi, \zeta(3), \zeta(5), \zeta(7), \ldots\}$$

Multiple zeta values

Definition

Let $r \in \mathbb{N}$, $k_1, \ldots, k_r \in \mathbb{N}$ be positive integers, with $k_r \geq 2$. We define

$$\zeta(k_1, \ldots, k_r) := \sum_{0<n_1<n_2<\ldots<n_r} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}}.$$

Theorem

The $\mathbb{Q}$-vector space spanned by multiple zeta values is a $\mathbb{Q}$-algebra.

As such, multiple zeta values satisfy many relations, e.g. $\zeta(1, 2) = \zeta(3)$.

The motivic Galois group

The Tannakian category $\mathcal{MT}(\mathbb{Z})$ of mixed Tate motives over $\mathrm{Spec}(\mathbb{Z})$ is equivalent to the category of finite dimensional representations of an affine group scheme $\mathrm{G}_{\mathcal{MT}(\mathbb{Z})}$, called its Galois group.

Theorem (Deligne [3])

$G_{\mathcal{MT}(\mathbb{Z})}$ decomposes as

$$G_{\mathcal{MT}(\mathbb{Z})} \cong \mathbb{G}_m \ltimes U_{\mathcal{MT}(\mathbb{Z})}$$

where $U_{\mathcal{MT}(\mathbb{Z})}$ is pro-unipotent with Lie algebra

$$\mathfrak{g}^{\mathfrak{m}} \cong Lie[\sigma_3, \sigma_5, \ldots].$$

The motivic Galois group

Theorem ([3])

For a $\mathbb{Q}$-algebra R,

$$G_{\mathcal{MT}(\mathbb{Z})}(R) \subset R\langle\langle e_0, e_1\rangle\rangle$$

consists of power series whose coefficients satisfy the motivic relations

Theorem (Brown [1])

(Regularised) MZVs satisfy the motivic relations.

Conjecture

The motivic relations describe all relations among multiple zeta values

Relations among MZVs

Associator
: MZVs give coefficients of a *Drinfeld associator*. This gives explicit relations and has connections to the Grothendieck Teichmüller group [4].

Double Shuffle
: Splitting the domain of summation in a product gives the stuffle relations. A similar process for an integral representation gives the shuffle relations [5]

Relations among MZVs

Associator MZVs give coefficients of a *Drinfeld associator*. This gives explicit relations and has connections to the Grothendieck Teichmüller group [4].

Double Shuffle Splitting the domain of summation in a product gives the stuffle relations. A similar process for an integral representation gives the shuffle relations [5]

Conjecture

The double shuffle and associator relations are equal to the motivic relations.

Examples

Example

$$\zeta(2)\zeta(3) = \sum_{m\geq 1}\sum_{n\geq 1}\frac{1}{m^2n^3}$$
$$= \sum_{m>n\geq 1}\frac{1}{m^2n^3} + \sum_{n>m\geq 1}\frac{1}{n^3m^2}$$
$$+ \sum_{m=n\geq 1}\frac{1}{n^5}$$
$$= \zeta(2,3) + \zeta(3,2) + \zeta(5)$$

Lie algebras of relations

To each set of relations we can associate an affine group scheme

$$\begin{array}{rc} \{\text{Double shuffle}\} & \mathsf{DMR} \\ & \bigcup \\ \{\text{Associator relations}\} & \mathsf{GT} \\ & \bigcup \\ \{\text{Motivic relations}\} & G_{\mathcal{MT}(\mathbb{Z})} \end{array}$$

By considering the associated Lie algebras, we can reduce the question of equality and explicit descriptions to one of linear algebra

Lie algebras of relations

$$\mathfrak{g}^{\mathfrak{m}} \quad \subset \quad \mathfrak{grt} \quad \subset \quad \mathfrak{dmr}_0 \quad \subset \quad \mathbb{Q}\langle e_0, e_1\rangle$$

Relations among the coefficients of elements of these Lie algebras describe relations among multiple zeta values modulo products

Lie algebras of relations

$$\mathfrak{g}^{\mathfrak{m}} \quad \subset \quad \mathfrak{grt} \quad \subset \quad \mathfrak{dmr}_0 \quad \subset \quad \mathbb{Q}\langle e_0, e_1\rangle$$

Relations among the coefficients of elements of these Lie algebras describe relations among multiple zeta values modulo products

Example

Modulo products

$$\zeta(2,3) + \zeta(3,2) + \zeta(5) = 0$$

Filtrations on multiple zeta values

Definition

Given a MZV $\zeta(k_1, \ldots, k_r)$, define its *weight* $k_1 + k_2 + \cdots + k_r$ and *depth* r.

Conjecture

MVZs are weight-graded.

Theorem

Motivic relations are weight graded.

Depth graded MZVS

Depth does not induce a grading, but we can consider the associated graded Lie algebra. Relations among its coefficients describe relations among MZVs modulo products and terms of lower depth.

Example

The double shuffle relations say that

$$\zeta(2,3)+\zeta(3,2)+\zeta(5)=\zeta(2)\zeta(3).$$

The depth graded version of this, modulo products, is

$$\zeta(2,3)+\zeta(3,2)=0.$$

Depth graded MZVs

Pro

- Graded $\{\sigma_{2k+1}\}$ have canonical representatives in $\mathbb{Q}\langle e_0, e_1\rangle$.
- Graded shuffle relations are much easier

Con

- Graded $\{\sigma_{2k+1}\}$ are no longer a generating set.
- Additional relations due to modular forms

Depth graded MZVs

Pro

- Graded $\{\sigma_{2k+1}\}$ have canonical representatives in $\mathbb{Q}\langle e_0, e_1\rangle$.
- Graded shuffle relations are much easier

Con

- Graded $\{\sigma_{2k+1}\}$ are no longer a generating set.
- Additional relations due to modular forms

Example ([2])

Modulo terms of depth three or higher,

$$\{\sigma_3, \sigma_9\} - 3\{\sigma_5, \sigma_7\} = 0.$$

Hence, there exist 'exceptional' generators.

The block filtration

A MZV can be identified uniquely with a noncommative monomial w in $\{e_0, e_1\}$.

$$\zeta(k_1, \ldots, k_r) \leftrightarrow e_1 e_0^{k_1 - 1} e_1 e_0^{k_2 - 1} \ldots e_1 e_0^{k_r - 1}$$

We often write $\zeta(w)$ for $\zeta(k_1, \ldots, k_r)$ under this identification. I introduce the following degree.

Definition

Define the *block degree* of this word $\deg_{\mathcal{B}}(w)$ to be the number of occurrences of a subsequence $e_i e_i$ for $i \in \{0, 1\}$ in $e_0 w e_1$.

This defines a unique factorisation into words of length $\ell_1, \ldots, \ell_{\deg_{\mathcal{B}}(w)+1}$.

The block filtration

Define

$$\mathsf{I}^{\mathfrak{m}}(\ell_1, \ldots, \ell_{\deg_{\mathcal{B}}(w)+1}) := (-1)^r \zeta(k_1, \ldots, k_r).$$

Definition

Define the block filtration on $\mathbb{Q}\langle e_0, e_1 \rangle$ by

$$B_n \mathbb{Q}\langle e_0, e_1 \rangle := \langle w \mid \deg_{\mathcal{B}}(w) \leq n \rangle_{\mathbb{Q}}.$$

This induces a filtration on the space of multiple zeta values

$$B_n \mathcal{Z} := \langle \mathsf{I}^{\mathfrak{m}}(\ell_1, \ldots, \ell_{m+1}) \mid m \leq n \rangle_{\mathbb{Q}}.$$

Block graded Lie algebras of relations

Definition

Define the block graded motivic Lie algebra

$$\mathfrak{bg} := \bigoplus_{n \geq 1} \mathcal{B}^n \mathfrak{g}^{\mathfrak{m}} / \mathcal{B}^{n+1} \mathfrak{g}^{\mathfrak{m}}.$$

Block graded Lie algebras of relations

Definition

Define the block graded motivic Lie algebra

$$\mathfrak{bg} := \bigoplus_{n \geq 1} \mathcal{B}^n \mathfrak{g}^{\mathfrak{m}} / \mathcal{B}^{n+1} \mathfrak{g}^{\mathfrak{m}}.$$

Theorem (K.)

$$\mathfrak{g}^{\mathfrak{m}} \cong \mathfrak{bg}$$

Block graded Lie algebras of relations

Definition

Define the block graded motivic Lie algebra

$$\mathfrak{bg} := \bigoplus_{n \geq 1} \mathcal{B}^n \mathfrak{g}^{\mathfrak{m}} / \mathcal{B}^{n+1} \mathfrak{g}^{\mathfrak{m}}.$$

Theorem (K.)

$$\mathfrak{g}^{\mathfrak{m}} \cong \mathfrak{bg}$$

Example

Modulo terms of lower block degree

$$\zeta(5) = 0.$$

The block graded Lie algebra

Pros

- No Lie algebraic information lost.
- Graded $\{\sigma_{2k+1}\}$ have canonical representatives.
- The Lie bracket is easier to compute.

.

Cons

- Known relations hard to grade.
- No known defining relations.

In my recent work, I provide severaly families of block graded relations, including complete set of relations describing $\mathcal{B}^1\mathfrak{g}^{\mathfrak{m}}/\mathcal{B}^2\mathfrak{g}^{\mathfrak{m}}$.

Block shuffle

Theorem (K.)

Then modulo products and terms of lower block degree

$$\sum_{\sigma \in Sh(k,n-k)} I^{\mathfrak{m}}(\ell_{\sigma^{-1}(1)}, \ldots, \ell_{\sigma^{-1}(n)}) = 0.$$

Corollary (Charlton's cyclic insertion conjecture)

$$\sum_{\sigma \in C_n} I^{\mathfrak{m}}(\ell_{\sigma^{-1}(1)}, \ldots, \ell_{\sigma^{-1}(n)}) = 0.$$

Block relations

Lemma (K.)

$$\mathfrak{bg} \hookrightarrow \bigoplus_{n \geq 0} \mathbb{Q}[x_1, \ldots, x_n].$$

Theorem (K.)

Identifying $\mathfrak{bg}$ *with its image, we can uniquely describe* $\mathfrak{bg}$ *in low block degree via explicit relations.*

Reduced block relations

Lemma

$$\mathfrak{bg} \hookrightarrow \bigoplus_{n \geq 0} \mathbb{Q}[x_1, \ldots, x_n].$$

We denote the image by $\mathfrak{rbg}$.

Reduced block relations

Lemma

$$\mathfrak{bg} \hookrightarrow \bigoplus_{n \geq 0} \mathbb{Q}[x_1, \ldots, x_n].$$

We denote the image by $\mathfrak{rbg}$.

Theorem (K.)

Let $r(x_1, \ldots, x_n) \in \mathfrak{rbg}$, and let $\sigma \in D_n$ be an element of the dihedral group. Then

$$r(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = sgn(\sigma)^n r(x_1, \ldots, x_n)$$

Reduced block relations

Theorem (K.)

Define the differential operator

$$D := \prod_{(\bullet_1,\ldots,\bullet_{n-1}) \in \{+,-\}^{n-1}} \frac{\partial}{\partial x_1} \bullet_1 \frac{\partial}{\partial x_2} \bullet_2 \cdots \bullet_{n-1} \frac{\partial}{\partial x_n}$$

Then $Dr = 0$ *for all* $r(x_1, \ldots, x_n) \in \mathfrak{rbg}$.

Reduced block relations

Theorem (K.)

Define the differential operator

$$D := \prod_{(\bullet_1,\ldots,\bullet_{n-1})\in\{+,-\}^{n-1}} \frac{\partial}{\partial x_1}\bullet_1\frac{\partial}{\partial x_2}\bullet_2\cdots\bullet_{n-1}\frac{\partial}{\partial x_n}$$

Then $Dr = 0$ *for all* $r(x_1,\ldots,x_n) \in \mathfrak{rbg}$*.*

Theorem (K.)

Along with two technical regularisation conditions, these relations uniquely describe $\mathfrak{bg}$ *in low block degree.*

Producing genuine relations

Given a defining relation in $\mathfrak{bg}$, can we lift it to a genuine motivic relation? We often can. For example, as a consequence of the block shuffle relation, we find the following.

Producing genuine relations

Given a defining relation in $\mathfrak{bg}$, can we lift it to a genuine motivic relation? We often can. For example, as a consequence of the block shuffle relation, we find the following.

Theorem (K.)

For any k, $n \geq 0$,

$$\zeta(\{2\}^k ⧢ \{1,3\}^n) = \frac{\pi^{4n+2k}\binom{2n+k}{k}}{(2n+1)(4n+2k+1)!}$$

where the left hand side is the sum over all shuffles of $(2,2,2,\ldots,2)$ *with* $(1,3,1,3,\ldots,1,3)$.

Producing genuine relations

Upcoming work due to Hirose and Sato gives the following lift of the block shuffle relation

Definition

Given tuples of positive integers, $(k_1, \ldots, k_m)$, $(l_1, \ldots, l_n)$, define recursively the formal sum

$$\begin{aligned}(k_1, \ldots, k_m) \hat{\sqcup} (l_1, \ldots, l_n) &= (k_1, (k_2, \ldots, k_m) \hat{\sqcup} (l_1, \ldots, l_n)) \\ &+ (l_1, (k_1, \ldots, k_m) \hat{\sqcup} (l_2, \ldots, l_n)) \\ &- L_{k_1+l_1}((k_2, \ldots, k_m) \hat{\sqcup} (l_2, \ldots, l_n))\end{aligned}$$

where

$$(k_1, \ldots, k_m) \hat{\sqcup} \emptyset = \emptyset \hat{\sqcup} (k_1, \ldots, k_m) = (k_1, \ldots, k_m)$$

and $L_s((k_1, \ldots, k_m)) = (k_1 + s, k_2, \ldots, k_m)$.

Producing genuine relations

Theorem (Hirose, Sato)

Extending $I^{\mathfrak{m}}$ linearly,

$$I^{\mathfrak{m}}((k_1, \ldots, k_m)\hat{\sqcup\!\sqcup}(l_1, \ldots, l_n) = 0$$

modulo products (up to some normalisation)

Producing genuine relations

Theorem (Hirose, Sato)

Extending $I^{\mathfrak{m}}$ linearly,

$$I^{\mathfrak{m}}((k_1, \ldots, k_m) \hat{ш} (l_1, \ldots, l_n) = 0$$

modulo products (up to some normalisation)

Theorem (K.)

For $\ell_1 + \ell_2 + \ell_3 = 2N + 2$

$$\begin{aligned} & I^{\mathfrak{m}}(\ell_1 - 1, \ell_2, \ell_3) + I^{\mathfrak{m}}(\ell_1 - 1, \ell_3, \ell_2) \\ & - I^{\mathfrak{m}}(\ell_1, \ell_2 - 1, \ell_3) - I^{\mathfrak{m}}(\ell_1, \ell_3 - 1, \ell_2) - \beta\zeta(2)^N \\ & = \sum_{\substack{r \leq s \\ 2r+2s+2 \leq 2N}} \alpha_{r,s}\zeta(2r+1)\zeta(2s+1)\zeta(2)^{N-r-s-1}. \end{aligned}$$

Thank you!

Questions?

References I

F. Brown.
Mixed Tate motives over $\mathbb{Z}$.
Annals of Math., 175(2):949 – 976, 2012.

F. Brown.
Depth-graded motivic multiple zeta values.
arXiv:1301.3053, 2013.

P. Deligne.
Le groupe fondamental de la droite projective moins trois points.
Galois groups over $\mathbb{Q}$, Math.Sci. Res. Inst. Publ, 16:79–297, 1989.

V.G. Drinfeld.
On quasitriangular quasi-Hopf algebras and on a groups that is closely associated with

References II

G. Racinet.
Double mélanges des polylogarithmes multiples aux racines de l'unité.
Publ. Math. Inst. Hautes tudes Sci., 95:185 – 231, 2002.