Relations and Filtrations of Multiple Zeta Values

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Definition

Let $r \in \mathcal{N}$, $k_1, \ldots, k_r \in \mathcal{N}$ be postive integers, with $k_r \geq 2$. We define

$$\zeta(k_1, \ldots, k_r) := \sum_{0<n_1<n_2<\ldots<n_r} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}}$$

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To a MZV, we can associate two quantities: *weight* and *depth*. The weight of $\zeta(k_1, \ldots, k_r)$ is defined to be $n_1 + n_2 + \cdots + n_r$, and the depth is defined to be r.

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$$w = e_1 e_0^{k_1 - 1} e_1 e_0^{k_2 - 1} e_1 \ldots e_1 e_0^{k_r - 1}$$

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$$w = e_1 e_0^{k_1 - 1} e_1 e_0^{k_2 - 1} e_1 \dots e_1 e_0^{k_r - 1}$$

and a differential form

$$\omega_w = \prod_{i=1}^{|w|} \frac{dt_i}{t_i - x_i}$$

where $x_i = n$ if $w_i = e_n$

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One can then show that

$$\zeta(k_1, \ldots, k_r) = (-1)^r \int_\Delta \omega_w$$

where we integrate over the simplex
$\Delta = \{(t_1, \ldots, t_{|w|}) | 0 \leq t_1 \leq t_2 \leq \ldots t_{|w|} \leq 1\}$

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Extending this definition by $\mathbb{Q}$-linearity, we obtain a map

$$\zeta : \mathbb{Q}\langle e_0, e_1 \rangle \to \mathbb{C}$$

The Double Shuffle Relations

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However, double shuffle relations are easier to work with, play well with the underlying motivic structure, and let us calculate upper bounds on the graded dimension of the $\mathbb{Q}$-span of MZVs

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$$\begin{aligned}\zeta(s)\zeta(t) &= \sum_{m>n\geq 1} + \sum_{n>m\geq 1} + \sum_{m=n\geq 1}\frac{1}{m^s n^t}\\ &= \zeta(t,s) + \zeta(s,t) + \zeta(s+t)\end{aligned}$$

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This is a *stuffle* relation, and holds for all MZVs.

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$$\zeta(2)\zeta(2) = \int_{0\le t_1\le t_2\le 1}\int_{0\le s_1\le s_2\le 1} \frac{dt_1\,dt_2\,ds_1\,ds_2}{(1-t_1)t_2(1-s_1)s_2}$$

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These are called the shuffle relations.

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The stuffle relations can similarly be described by $\Delta^*\Phi = \Phi \otimes \Phi$ for a coproduct Δ^*

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Definition

Define DMR_0 to be the scheme such that, for a $\mathbb{Q}$-algebra R,

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Theorem (Racinet, 2002)

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Theorem (Drinfel'd 1991, Furusho 2008)

$DMR_0(\mathbb{Q})$ is non empty

Racinet's Lie Algebra

We now move from $\mathrm{DMR}_0(\mathbb{Q})$ to the Lie algebra $\mathfrak{dmr}_0$. This amounts to considering the double shuffle relations modulo products, or primitive elements of $\mathbb{Q}\langle e_0, e_1\rangle$.

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$$\begin{aligned}\circ : \mathbb{Q}\langle e_0, e_1\rangle \otimes \mathbb{Q}\langle e_0, e_1\rangle &\to \mathbb{Q}\langle e_0, e_1\rangle \\ u \otimes e_0^n e_1 v &\mapsto e_0^n u e_1 v + e_0^n e_1 u^* v + e_0^n e_1 (u \circ v)\end{aligned}$$

where $(u_1 u_2 \ldots u_r)^* = (-1)^r u_r \ldots u_1$ is the antipode map.

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where $(u_1 u_2 \dots u_r)^* = (-1)^r u_r \dots u_1$ is the antipode map.
Drinfel'd shows the existence of a Lie algebra

$$\mathfrak{g} = \mathsf{Lie}(\sigma_{2k+1}) \hookrightarrow \mathfrak{dmr}_0$$

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Next we have the depth filtration.This is definitely *not* a grading. As such, we can consider the associated graded of $\mathfrak{dmr}_0$ with respect to the depth filtration.

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- Depth is asymmetric: depth is not invariant under the change of variables $x \mapsto 1 - x$, but the numerical values are.
- "Depth graded associator equations" seem impossible to write down.
- There are obstructions due to modular forms: additional relations and additional generators

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The category of mixed Tate motives MTM is isomorphic as a Tannakian category to the motivic fundemental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

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Corollary

The period of a mixed Tate motive is a $\mathbb{Q}$-linear combination of MZVs.

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Theorem

MTM is equivalent to the category of representations of a group scheme $G \equiv \mathbb{G}_m \ltimes U$, where U is prounipotent. This group acts on realisations of mixed Tate motives

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Instead of considering this action, we instead consider a coaction
$\Delta : \mathcal{O}({}_0\Pi_1) \to \mathcal{O}(G) \otimes \mathcal{O}({}_0\Pi_1)$

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The straight line path dch maps $\mathcal{O}({}_0\Pi_1)$ *to the* $\mathbb{Q}$*-span of MZVs.*

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We define $\mathcal{H}$ to be the quotient of $\mathcal{O}({}_0\Pi_1)$ by the "motivic" kernel of this map. This is the space of motivic multiple zeta values, spanned by $I(a_0; a_1, \dots, a_n; a_{n+1}),\ a_i \in \{0, 1\}$.

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And a coaction

$$\Delta : \mathcal{H} \to \mathcal{H}/(I(0; 10; 1)) \otimes \mathcal{H}$$

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Let $X \in \mathcal{H}$ be of weight N. If $\Delta X = X \otimes 1 + 1 \otimes X$, then $X \in \mathbb{Q}\zeta^{\mathfrak{m}}(N)$.

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This is a very powerful tool. Additionally, all known relations among MZVs are "stable" under the coaction, so it would make sense to consider filtrations that are "stable" under the coaction.

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In particular, we will consider the *coradical* filtration.

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Definition

For a vector space $V \subset \mathbb{Q}\langle e_0, e_1 \rangle$, define $B_n V = V \cap \mathrm{Span}_{\mathbb{Q}}\{w | \deg_{\mathcal{B}} \geq n\}$

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Theorem

The Ihara action $\circ : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ *is a graded product i.e*

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Corollary

The projection $\mathfrak{g} \to \mathfrak{bg}$ *commutes with the Ihara action.*

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Thus we can write

$$I(0; w; 1) = I_{bl}(l_1^{(i_1)}, \ldots, l_k^{(i_k)})$$

where $l^{(i)} \in \mathbb{N}$ represents the alternating binary string of length l beginning with i.

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where $l^{(i)} \in \mathbb{N}$ represents the alternating binary string of length l beginning with i. Similarly, we can write

$$e_0 w e_1 = z_{n_1}^{i_1} \dots z_{n_k}^{i_k}$$

where $z_n^i \in \mathbb{Q}\langle e_0, e_1 \rangle$ denotes the alternating word of length n beginning with e_i

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Conjecture (Charlton, 2017)

$$\sum_{\sigma \in C_n} I_{bl}(l_{\sigma(1)}, \ldots, l_{\sigma(k)}) = 0$$

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Proof of this theorem relies heavily on explicit calcuations with the motivic coaction.

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This proves Charlton's conjecture in low block degree. Conjecturally, we expect this to hold modulo products, however, that remains a work in progress.

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Proof.

Let $\sigma \in \mathfrak{bg}_1$, and consider it's projection p_σ in $\mathbb{Q}[x_1, x_2]$, where for $e_0 w e_1 = z_1 z_2$,

$$w \mapsto x_1^{|z_1|} x_2^{|z_2|}$$

It is well known that $p_\sigma(x_1, x_2) + p_\sigma(x_2, x_1) = 0$, proving the result in block degree one. Explicit computation of the Ihara action in terms of these polynomials shows that this property is preserved.

$$\sum_{\sigma \in \mathrm{C}_n} p(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma_n}) = 0$$

$\square$

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This provides a whole new class of relations, that again seem to hold numerically in the original Lie algebra. Even with these relations, we cannot uniquely describe $\mathfrak{bg}$ as via equations.

Block Graded Double Shuffle

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Theorem (Block Graded Shuffle, K.)

Considering $\mathfrak{bg} \subset \mathbb{Q}\langle e_0, e_1\rangle$*, and denoting by* $\pi_1 : \mathbb{Q}\langle e_0, e_1\rangle \to \mathbb{Q}e_0 \oplus \mathbb{Q}e_1$ *the natural projection map, we have that* $(\pi_1 \otimes id) \circ \Delta(\mathfrak{bg}) = 0$

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Theorem (Block Graded Stuffle, K.)

Let $(m_1, \ldots, m_k)$, $(n_1, \ldots, n_l)$ be two sequences of integers, with $m_i, n_i > 1$ and $k \leq l$. Define $m_{k+1} = \ldots = m_l = 0$. Then

$$\sum_{\sigma \in Sh_{k,l-k}} \zeta_{bl}(m_{\sigma(1)} + n_1, \ldots, m_{\sigma(l)} + n_l = 0$$

considered modulo products and terms of lower degree.

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This can be extended to all multiple zeta values

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Thank you! Questions?