A Discussion of the Kadison-Singer Problem

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1. Introduction

The Kadison-Singer Problem was a major option problem in operator theory, formulated in 1959, until it's recent solution in 2013, found by Adam Marcus, Daniel Spielman and Nikhil Srivastava. It's solution was very exciting, as many equivalent problems have been formulated over the years, in areas including operator theory, discrepancy theory, algebraic geometry and more.
It is, however, in quantum mechanics that the Kadison-Singer problem finds its motivation. In his 1930 book, "The Principals of Quantum Mechanics" [5], Paul Dirac discussed how to find a "representation", or and orthonormal basis, for observable quantities in quantum mechanical experiments:

To introduce a representation in practice

1. **We look for observables which we would like to have diagonal, either because we are interested in their probabilities or for reasons of mathematical simplicity;**
2. **We must see that they all commute - a necessary condition since diagonal matrices always commute;**
3. **We then see that they form a complete commuting set, and if not we add some more commuting observables to them to make them into a complete commuting set;**
4. **We set up an orthogonal representation with this complete commuting set diagonal.**

The representation is the completely determined except for the arbitrary phase factors.

Unfortunately, Dirac was mistaken in this fianl statement, and this is precisely the Kadison-Singer Problem: by considering quantum observables as commutating operators on a Hilbert space, and representations as pure states on this space of operators, the question becomes:
When can we uniquely extend a pure state on a maximal abelian self-adjoint subalgebra?

In their 1959 paper [6], Kadison and Singer showed that, for certain algebras, including some arising in quantum field theory, the extension is not unique. They also felt it unlikely that is held for all algebras arising in quantum mechanics generally, but, fortunately for Dirac, the solution of Marcus, Spielman and Srivastava gives a postive result for discrete algebras [7]

3.The Paving Problem

One of the most common discussed equivalent formulations of the Kadison-Singer Problem is Anderson's Infinite Dimensional Paving Conjecture, introduced in [1]. The statement is as follows:

Theorem. *For every $\epsilon > 0$, there exists $r \in \mathbb{N}$ such that, for every self adjoint $H \in \mathcal{B}(\ell_2)$ with $D(H) = 0$, there exists a partition $\{A_1, A_2, \ldots, A_r\}$ of $\mathbb{N}$ such that $\|P_{A_i} H P_{A_i}\| \leq \epsilon\|H\|$ for all $1 \leq i \leq r$.*

Here we have that $P_A, A \subset \mathbb{N}$ is a **projection** operator, and $D(H)$ is the **diagonal** of H.
It can be shown that the Paving Problem is equivalent to the Kadison-Singer Problem, and also to a finite dimensional Paving Problem [2]

Theorem. *For every $\epsilon > 0$, there exists $r \in \mathbb{N}$ such that, for every self adjoint $H \in \mathcal{B}(\mathbb{C}^n)$ with $D(H) = 0$, there exists a partition $\{A_1, A_2, \ldots, A_r\}$ of $\{1, 2, \ldots, n\}$ such that $\|P_{A_i} H P_{A_i}\| \leq \epsilon\|H\|$ for all $1 \leq i \leq r$.*

5.The Bourgain Tzafriri Problem

Another formulation of the Kadison-Singer Problem arose from a theorem of Bourgain and Tzafriri in a paper from 1987, called the **restricted invertibility principle**.

Theorem. *There exist universal constants $A, c > 0$, so that, whenever $T : \ell_2^n \to \ell_2^n$ is a linear operator with $\|Te_i\| = 1$ for $1 \leq i \leq n$, then there exists a subset $S \subset \{1, 2, \ldots, n\}$ of cardinality $|S| \geq \frac{cn}{\|T\|^2}$ so that, for all $1 \leq j \leq n$ and all choice of scalars $\{a_j\}_{j\in S}$*

$$\|\sum_{j\in S} a_j Te_j\|^2 \geq A \sum_{j\in S} |a_j|^2,$$

It received a great deal of attention, leading to the following conjecture, which was shown to be equivalent to the Kadison-Singer Problem in [3]

Theorem. *There are universal constants $A > 0$ so that for every $B > 1$ there exists $r_B \in \mathbb{N}$ such that, whenever $T : \ell_2^n \to \ell_2^n$ is a linear operator with $\|Te_i\| = 1$ for $1 \leq i \leq n$, then there exists a partition $A_1, \ldots A_{r_B}$ of $\{1, 2, \ldots, n\}$ so that, for all $j = 1, 2, \ldots, n$ and all choices of scalars $\{a_j\}_{j\in S}$, we have*

$$\|\sum_{i\in A_j} a_i Te_i\|^2 \geq A \sum_{i\in A_j} |a_i|^2.$$

This is known as the **strong** form the the conjecture. In the weaker formulation, the constant A is replaced by a function $A(\|T\|)$. It was shown that both formulations were indeed equivalent in [2]

7. Proving the Result

KS_2 was shown in 2014 by Adam Marcus, Daniel Spielman and Nikhil Srivastava [7], using techniques from linear algebra and random matrix theory. In particular, they introduced new techniques involving the following, notoriously troublesome, representation of the operator norm

$$\|A\|_{op} = \mathrm{maxroot}(p_A)$$

where p_A is the characteristic polynomial of A and $\mathrm{maxroot}(p)$ is the largest real root of a non-zero polynomial p. Using these techniques to analyse certain families of random matrices and interlacing families of their characteristic polynomials to prove the following theorem:

Theorem. *If $\epsilon > 0$ and $v_1, v_2, \ldots, v_n$ are independent random vectors in $\mathbb{C}^k$ with finite support such that*

$$\sum_{i=1}^{n} \mathbb{E} v_i v_i^* = \mathbb{I}$$

and

$$\mathbb{E}\|v_i\|^2 \leq \epsilon \text{ for all } 1 \leq i \leq n$$

then

$$\mathbb{P}\left[\left\|\sum_{i=1}^{n} v_i v_i^*\right\| \leq (1 + \sqrt{\epsilon})^2\right] > 0.$$

Here $\mathbb{P}[X]$ represents the **probability** of X and $\mathbb{E}X$ represents the **mean**. KS_2 with $N = 18$ quickly follows as a corollary, resolving the Kadison-Singer Problem.

2. Kadison-Singer in Operator Theory

The original formulation of the Kadison-Singer Problem arises in operator theory. We consider a subalgebra of $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on a separable Hilbert space, which we endow with the structure of a C^*-algebra as is standard [8]. In particular, we consider a **maximal abelian self-adjoint** subalgebra, or a MASA.
We then look at **pure states** on this MASA, i.e norm 1 linear maps that cannot be written as a non-trivial convex combination of two different states, and extensions of these states to the entire space.
The precise statement of the Kadison-Singer Problem is them

Theorem. *Let $\mathcal{A}$ be a discrete MASA of $\mathcal{B}(\mathcal{H})$, the algebra of bounded linear operators on a separable Hilbert space. Let $\rho : \mathcal{A} \to \mathbb{C}$ be a pure state on that subalgebra. Then there exists a pure state extension $\rho' : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ of ρ that is unique.*

This result is reasonably straight forward if $H \simeq \mathbb{C}^n$, but in the case of infinite dimensional spaces, proved extremely challenging.

4. The Feichtinger Conjecture and Frame Theory

The Feichtinger Conjecture is a problem in frame theory, shown to be equivalent to the Kadison-Singer Problem in [4]. A **bounded frame** is a family of vectors $\{v_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ such that there exist constants $0 < A \leq B < \infty$ such that, for all $u \in \mathcal{H}$

$$A\|u\|^2 \leq \sum_{i\in I} |\langle u, v_i\rangle|^2 \leq B\|u\|^2.$$

A special class of frame is the **Riesz basic sequence**, which is a family of vectors $\{v_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ for which there exist $A, B > 0$ such that, for all scalar sequences $\{a_i\}_{i\in I}$ we have:

$$A \sum_{i\in I} |a_i|^2 \leq \|\sum_{i\in I} a_i v_i\|^2 \leq B \sum_{i\in I} |a_i|^2.$$

The Feichtinger conjecture ties these together neatly:

Theorem. *Every bounded frame is a finite union of Riesz basic sequences*

We also have a similar conjecture, called the R_ϵ conjecture, introduced in [3], and later shown to also be equivalent to the Kadison-Singer Problem. It introduced ϵ-Riesz basic sequences, where we replace (A, B) by $(1 - \epsilon, 1 + \epsilon)$

Theorem. *For every $\epsilon > 0$, every Riesz basic sequence consisting of vectors with unit norm is a finite union of ϵ-Riesz basic sequences*

6. The KS_r Conjecture

In his 2004 paper, [9], Weaver introduced a formulation of the Kadison-Singer Problem in discrepancy theory, which proved to be the key to the solution. The statement is as follows:

Theorem. *There exist universal constants $N \geq 2$ and $\epsilon > 0$ such that the following holds. Let $v_1, v_2, \ldots, v_n \in \mathbb{C}^k$ satisfy $\|v_i\| \leq 1$ for all $1 \leq i \leq n$ and suppose*

$$\sum_{i=1}^{n} |\langle u, v_i\rangle|^2 \leq N$$

for every unit vector $u \in \mathbb{C}^k$. Then there exists a partition $A_1, A_2, \ldots A_r$ of $\{1, 2, \ldots, n\}$ such that

$$\sum_{i\in A_j} |\langle u, v_i\rangle|^2 \leq N - \epsilon$$

for every unit vector $u \in \mathbb{C}^k$ and all $1 \leq j \leq r$.

Weaver also showed the following reformulation of KS_2, the case proven in [7].

Theorem. *There exist universal constants $N \geq 2$ and $\epsilon > 0$ such that the following holds. Let $v_1, v_2, \ldots, v_n \in \mathbb{C}^k$ satisfy $\|v_i\| \leq 1$ for all $1 \leq i \leq n$ and suppose*

$$\sum_{i=1}^{n} |\langle u, v_i\rangle|^2 \leq N$$

for every unit vector $u \in \mathbb{C}^k$. Then there some choice of signs such that

$$\sum_{i=1}^{n} \pm|\langle u, v_i\rangle|^2 \leq N - \epsilon$$

for every unit vector $u \in \mathbb{C}^k$.

8. Open Problems

Despite the resolution of the Kadison-Singer Problem, there still remains work to be done in this area. The solution of Marcus, Spielman and Srivastava is unlikely to be the optimal one, leaving several open problems.

Problem. *Can the constants in any of the formulations be improved?*

Problem. *Can every unit norm 2-tight frame be partitioned into three subsets, each of which are Riesz basic sequences, with Riesz bounds independent of the dimension of the ambient space?*

Problem. *Is there an implementable algorithm for proving the Paving Conjecture?*

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