A Discussion of the Kadison-Singer Conjecture

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Abstract

The Kadison-Singer problem is a major result in operator theory, recently proven by Marcus, Spielman and Srivastava. Since its formulation in 1959, it has generated a significant amount of interest, and much work has been done on it. It has been tied to problems in frame theory, harmonic analysis, quantum mechanics and has dozens of equivalent formulations. In this project, we shall discuss briefly the history of the problem, before discussing the problem itself, five equivalent formulations and sketching the proof. We then mention some of its implications and some remaining related open problems.

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1 Introduction

The 1959 Kadison-Singer problem was a major open problem in operator theory until quite recently, when in 2013 a solution was provided by Adam Marcus, Daniel A. Spielman and Nikhil Srivastava. Between the original formulation of the problem and its solution, many equivalent problems were formulated, in many fields of mathematics, including operator theory, discrepancy theory and algebraic geometry. In the following discussion, we shall introduce several of these equivalent formulations and briefly discuss the solution.

Let us begin by stating the original statement of the Kadison-Singer problem:

Problem 1.1. *Let $\mathcal{A}$ be a separable Hilbert space. Does every pure state on a M.A.S.A of the algebra of bounded linear operators $\mathcal{B}(\mathcal{A})$ have a unique extension to a pure state on $\mathcal{B}(\mathcal{A})$*

To those familiar with quantum mechanics, this idea of extensions of states should be familiar. The problem is believed to have come from a issue which Paul Dirac addressed in his book [20]. He wanted to find a "representation", or an orthonormal basis for observable quantities:

> **To introduce a representation in practice**
>
> 1. **We look for observables which we would like to have diagonal, either because we are interested in their probabilities or for reasons of mathematical simplicity;**
> 2. **We must see that they all commute - a necessary condition since diagonal matrices always commute;**
> 3. **We then see that they form a complete commuting set, and if not we add some more commuting observables to them to make them into a complete commuting set;**
> 4. **We set up an orthogonal representation with this complete commuting set diagonal.**
>
> **The representation is the completely determined except for the arbitrary phase factors.**

Unfortunately, Dirac was mistaken in this final statement, and this is the Kadison-Singer problem: we can consider quantum observables as commuting elements of $\mathcal{B}(\mathcal{A})$ and these representations as pure states. In their 1959 paper [26], Kadison and Singer showed that for certain continuous MASAs, including some arising in quantum field theory, the extension is not unique. While they were careful not to state it as a conjecture, they also stated that they felt it unlikely that it held in all discrete MASAs. Fortunately for Dirac, the solution of Marcus, Spielman and Srivastava gave a positive result in the case of discrete algebras [28].

2 Background

We will now develop the background needed for the standard statement of the Kadison-Singer problem. For the majority of this section, no specific reference will be mentioned, as any textbook on C^*-algebras will suffice. In particular, the author found [29] and [35] useful introductions. For the purposes of this discussion, we choose $\mathbb{C}$ as the ground field.

2.1 Banach Spaces and Algebras of Operators

Definition 2.1. A ***Banach space*** is a vector space equipped with a norm for which that space is complete with respect to the norm-induced metric.

Definition 2.2. A ***Hilbert space*** is a vector space equipped with an inner product, such that the space is a Banach space with the norm $\|v\| = \sqrt{< v, v >}$.

Definition 2.3. A linear operator $A : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is called bounded if there exists a $\lambda > 0$ such that $\|Ax\| \leq \lambda \|x\|$ for all $x \in \mathcal{H}$. We denote the space of linear operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$.
We can define a topology on $\mathcal{B}(\mathcal{H})$, called the ***weak-* topology***, which is the weakest topology such that all elements of a space called the *predual* are continuous. As it is not necessary for our purposes, we shall not elaborate on this.

Definition 2.4. A ***Banach algebra*** is a Banach space $\mathcal{B}$ equipped with a bilinear map $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$, $(x, y) \mapsto xy$ such that $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in \mathcal{B}$. We call a Banach algebra ***unital*** if there exists $\mathbb{I} \in \mathcal{B}$ such that $\|\mathbb{I}\| = 1$ and $x\mathbb{I} = \mathbb{I}x = x$ for all $x \in \mathcal{B}$.

Definition 2.5. A ***state*** on a unital Banach algebra $\mathcal{B}$ is a non-zero linear map $f : \mathcal{B} \to \mathbb{C}$ such that $\|f\| = f(\mathbb{I}) = 1$.

We are particularly concerned with states on a special class of algebras, called C^*-algebras.

Definition 2.6. A C^*-algebra is a complex associative Banach algebra $\mathcal{A}$ with a map $* : \mathcal{A} \to \mathcal{A}$ satisfying the following:

1. $x^{**} = (x^*)^* = x \forall x \in \mathcal{A}$
2. $(x + y)^* = x^* + y^* \forall x, y \in \mathcal{A}$
3. $(xy)^* = y^* x^* \forall x, y \in \mathcal{A}$
4. $(\lambda x)^* = \bar{\lambda} x^* \forall \lambda \in \mathbb{C}, x \in \mathcal{A}$
5. $\|x^* x\| = \|x\|\|x^*\|$

We call an element $A \in \mathcal{A}$ ***self-adjoint*** if $A = A^*$.
We call $\mathcal{A}$ ***unital*** if there exists $\mathbb{I} \in \mathcal{A}$ such that $A\mathbb{I} = \mathbb{I}A = A \forall A \in \mathcal{A}$.

It can be shown that, for any Hilbert space $\mathcal{A}$, that $\mathcal{B}(\mathcal{A})$ is a C^* algebra.

Definition 2.7. A ***linear functional*** on a Banach space $\mathcal{B}$ is a map $f : \mathcal{B} \to \mathbb{C}$, such that

$$f(\alpha A + \beta B) = \alpha f(A) + \beta f(B) \forall \alpha, \beta \in \mathbb{C}, A, B \in \mathcal{B}.$$

We can define a unique class of linear functionals on C^*-algebras, generalizing positive definite transformations of finite dimensional vector spaces.

Definition 2.8. If $\mathcal{A}$ is a C^*-algebra, and element $A \in \mathcal{A}$ is called ***positive*** if $A = B^*B$ for some $B \in \mathcal{A}$. Note that if A is positive, A is self-adjoint.

In the case of $n \times n$ complex matrices, this is equivalent to A is Hermitian and positive semidefinite.

Positivity induces a partial order on $\mathcal{A}$ defined by

$$A \succ B \Leftrightarrow A - B \text{ is positive.}$$

Definition 2.9. We call a linear functional $f : \mathcal{A} \to \mathbb{C}$ ***positive*** if $f(A) \in \{x \in \mathbb{R} : x \geq 0\}$ for all positive A.

Lemma 2.10. *If $f : \mathcal{A} \to \mathbb{C}$ is a positive linear functional, f preserves the order of $\mathcal{A}$, i.e $A \succ B \Rightarrow f(A) \geq f(B)$.*

Proof.

$$\begin{aligned} A \succ B &\Rightarrow A - B \text{ positive} \\ &\Rightarrow f(A - B) \geq 0 \\ &\Rightarrow f(A) - f(B) \geq 0 \\ &\Rightarrow f(A) \geq f(B) \end{aligned}$$

□

Example 2.11. If we consider $\mathbb{M}_n = \{(a_{ij})_{i,j=1}^n : a_{ij} \in \mathbb{C}\}$ is the space of complex $n \times n$ matrices, then $f : \mathbb{M}_n \to \mathbb{C}$ is positive if and only if there exists a positive semidefinite matrix B such that $f(A) = tr(AB) \ \forall \ A \in \mathbb{M}_n$.

We use positivity to refine the definition of states and pure states in the case of C^*-algebras, the objects of interest in the Kadison-Singer problem.

Proposition 2.12. *If $\mathcal{A}$ is a unital C^*-algebra, then linear functional f is a state if and only if f is positive and $f(\mathbb{I}) = 1$.*

Definition 2.13. A ***pure state*** is a state which cannot be written as a non trivial convex combination of two different states.

We can now begin to understand the statement of the problem.

2.2 Kadison-Singer in Finite Dimensions

Before diving into the remaining definitions required, we shall briefly illustrate the problem in a finite dimensional case. While mathematically uninteresting, it serves as an illuminating exercise nevertheless.

In two dimensions, the Kadison-Singer problem can be stated as:

> "Does every pure state on the space of bounded diagonal operators on $\mathbb{C}^2$ have a unique extension to the space of bounded operators?"

Here, "bounded operators" is simply the space of 2×2 matrices over the complex numbers, $\mathbb{M}^{2\times 2}$. A state is a linear map $f : \mathbb{M}^{2\times 2} \to \mathbb{C}$ such that:

- $f(\mathbb{I}) = 1$
- $f(M) \in \{x \in \mathbb{R}, x \geq 0\}$ whenever M is positive semidefinite.

The space of diagonal operators is

$$\mathbb{D}_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \ a, d \in \mathbb{C} \right\}$$

A state on this space can be described by $f(M) = f(a, b) = \alpha a + \delta d$ where α, δ must satisfy:

- $\delta = 1 - \alpha$
- $\alpha \in [0, 1]$

In order for a state to be pure, we must have that it is not a non trivial convex combination of two different states, so we find that the only pure states on $\mathbb{D}_2$ are $f(a, d) = a$ and $f(a, d) = d$.

The question now is, can we extend these uniquely to a pure state on $\mathbb{C}^{2\times 2}$.

$$\mathbb{M}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ a, b, c, d \in \mathbb{C} \right\}$$

A state on this space is a map $g : \mathbb{C}^{2\times 2} \to \mathbb{C}$ with $g(M) = g(a, b, c, d) = \alpha a + \beta b + \gamma c + \delta d$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. It must also satisfy:

- $g(I) = 1$.
- $g(M)$ is real and non-negative whenever M is Hermitian, positive semidefinite.

We call g an ***extension*** of f if $g(a, 0, 0, d) = f(a, d)$ for all $M \in \mathbb{D}^2$. There is an obvious extension in which we simply define $g(a, b, c, d) := f(a, d)$.

For a state which is not pure, such as $f(a,d) = \frac{a+b}{2}$, this extension is not unique, as both

$$g(a,b,c,d) = \frac{a+b+c+d}{2}$$
$$g(a,b,c,d) = \frac{a+d}{2}$$

are extensions of f to states on $\mathbb{C}^{2\times 2}$.

However, for a pure state, the extension will be unique. Consider the pure state on $\mathbb{D}^2$ given by $f(a,d) = d$. If $g(a,b,c,d) = \alpha a + \beta b + \gamma c + \delta d$ is an extension of f, we must have

$$\alpha = 0$$
$$\beta = \bar{\gamma}$$
$$\delta = 1$$

in order to satisfy $g(a,0,0,d) = f(a,d)$ and $g(M) \in \mathbb{R}$. In order that g be positive, we require that the matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ be positive semidefinite. Thus

$$\begin{aligned} \alpha\delta - \beta\gamma \geq 0 &\Leftrightarrow -\gamma\bar{\gamma} \geq 0 \\ &\Leftrightarrow \gamma = 0 \end{aligned}$$

Thus the only extension of $f(a,d) = d$ to $\mathbb{C}^{2\times 2}$ is $g(a,b,c,d) = d$. We conclude similarly that $f(a,d) = a$ has a unique extension. Thus, over $\mathbb{C}^{2\times 2}$, the Kadison-Singer problem is true.

2.3 ℓ_∞ and Ultrafilters

In this section we briefly introduce several concepts, that, while not critical to the statement of the Kadison-Singer Problem in its standard presentation, prove useful in understanding many of the equivalent formulations. For sake of brevity, we shall omit any proofs that do not enlighten some other aspect of the theory. For more detail on the space ℓ_2, we recommend [19], or any introductory functional analysis text. For more information regarding ultrafilters, we recommend that the reader try [24].

Definition 2.14.

$$\ell_2 := \left\{ \{a_n\}_{n=1}^{\infty} : a_n \in \mathbb{C}\ \forall n, \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

Lemma 2.15. *ℓ_2 is a Hilbert space with norm induced by the bilinear form*

$$\langle \{a_n\}, \{b_n\} \rangle := \sum_{n=1}^{\infty} |a_n b_n|$$

and basis $\{e_i\}_{i=1}^{\infty}$ where

$$(e_i) = \{\delta_j^i\}_{j=1}^{\infty}$$

Definition 2.16. A ***diagonal operator*** on ℓ_2 is a linear map $D : \ell_2 \to \ell_2$ for which there exists $c_1, c_2, \ldots \in \mathbb{C}$ with $D(a_1, a_2, \ldots) = (c_1 a_1, c_2 a_2, \ldots)$ for every $(a_1, a_2, \ldots) \in \ell_2$.

We denote by $\mathbb{D}^2$ the space of bounded diagonal operators on ℓ_2.

Proposition 2.17. *$\mathbb{D}^2$ and ℓ_∞ are isometrically isomorphic.*

Definition 2.18. A ***projection*** in $\mathbb{D}^2$ is an operator $P_A \in \mathbb{D}^2$, $A \subset \mathbb{N}$ such that

$$\langle P_A(e_i), e_i \rangle = \begin{cases} 0 & i \in A \\ 1 & i \notin A \end{cases}$$

We now define a MASA, and a discrete MASA, of which $\mathbb{D}^2$ is an example.

Definition 2.19. A ***maximal Abelian self-adjoint subalgebra***, or ***MASA***, of a C^*-algebra $\mathcal{A}$, is a subalgebra $B \subset \mathcal{A}$ that is Abelian, closed under the $*$ operator and is not contained in any larger Abelain self-adjoint subalgebra.

Definition 2.20. For a Hilbert space $\mathcal{H}$, an element $P \in \mathcal{B}(\mathcal{H})$ is called ***minimal projection*** if $P = P^* = P^2$ and $\dim P(\mathcal{H}) = 1$. A MASA $B \subset \mathcal{B}(\mathcal{H})$ is called ***discrete*** if it is the weak-* closure of the minimal projections it contains.

Example 2.21. $\mathbb{D}^2$ is discrete, by definition, and will be our main example of a discrete MASA.
The space $\{M_f : f \in \mathrm{L}^\infty[0,1]\} \subset \mathcal{B}(\mathrm{L}^2[0,1]$ is not, where

$$\mathrm{L}^2[0,1] := \{f : [0,1] \to \mathbb{C} : f \text{ measurable}, \int_0^1 |f|^2 < \infty\}$$

$$\mathrm{L}^\infty[0,1] := \{f : [0,1] \to \mathbb{C} : |f| \text{ is measurable and bounded}\}$$

and $M_f(g) = fg$. To be more precise, $\mathrm{L}^p[0,1]$ is the space of equivalence classes of such functions, where $f \equiv g$ if $f = g$ almost everywhere.

Proposition 2.22 ([35] Ex. 5.15)**.** *All discrete MASAs in $\mathcal{B}(\ell_2)$ are unitarily equivalent to $\mathbb{D}^2$.*

Definition 2.23. Let X be a non empty set. A ***filter*** on X is a collection $\mathcal{F} \subset 2^X$ with the following properties:

- $X \in \mathcal{F}$.
- $\emptyset \notin \mathcal{F}$.
- If $A \in \mathcal{F}$ and $B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.
- If $A \in \mathcal{F}$ and $A \subset B$, then $B \in \mathcal{F}$.

We call a filter $\mathcal{F}$ an ***ultrafilter*** if in addition:

- For every $A \subset X$, exactly one of A or $X \setminus A$ is in $\mathcal{F}$.

Example 2.24. Choose $x \in X$ and define $\mathcal{U}_x := \{A \subset X : x \in A\}$. This clearly satisfies the definition of an ultrafilter. Such ultrafilters are called ***principal ultrafilters***.

If X is finite, then every ultrafilter is principal. Indeed, even for infinite X, if $\mathcal{U}$ is an ultrafilter containing a finite set, $\mathcal{U}$ is a principal ultrafilter. However, if we assume the axiom of choice, we can construct non-principal, or ***free*** ultrafilters for infinite X. As we will see in Theorem 2.32, this gives rise to 'non-principal' pure states, which makes the Kadison-Singer Problem significantly more difficult than the case presented in Section 2.2.

Lemma 2.25. *A filter $\mathcal{F} \subset 2^X$ is an ultrafilter if and only if it is a maximal filter with respect to inclusion of collections of sets.*

The proof of this statement follows easily by assuming the contrary and applying the definition of an ultrafilter.

Lemma 2.26. *Let $\mathcal{U} \subset 2^X$ be an ultrafilter and suppose $A = \cup_{i=1}^{k} A_i \in \mathcal{U}$. Then $A_i \in \mathcal{U}$ for some i.*

Proof. As the case where $k = 1$ is obvious, we shall consider $k \geq 2$. If $A_k \in \mathcal{U}$ we are done. Otherwise $X \setminus A_k = \cup_{i=1}^{k-1} A_i \in \mathcal{U}$. Thus $A \setminus A_k = A \cap (X \setminus A_k) \in \mathcal{U}$. But $A \setminus A_k \subset \cup_{i=1}^{k-1} A_i$, so $\cup_{i=1}^{k-1} A_i \in \mathcal{U}$. By induction, $\mathcal{U}$ contains some set A_i, $1 \leq i \leq k-1$. □

The collection of all ultrafilters of $\mathbb{N}$ is denoted $\beta(\mathbb{N})$. This can in fact be identified with the Stone-Čech compactification of $\mathbb{N}$ [32], [18]. As we will soon see $\beta(\mathbb{N})$ can be identified with the space of pure states on $\mathbb{D}^2$.

Definition 2.27. Define $\beta(\mathbb{N}) := \{\mathcal{U} : \mathcal{U} \text{ is an ultrafilter on } \mathbb{N}\}$ and let

$$\hat{A} := \{\mathcal{U} \in \beta(\mathbb{N}) : A \in \mathcal{U}\} \ \forall A \subset \mathbb{N}$$
$$\mathcal{A} := \{\hat{A} : A \subset \mathbb{N}\}$$

Lemma 2.28. *$\mathcal{A}$ forms a base for a topology on $\beta(\mathbb{N})$.*

Proof. Clearly $\cup_{\hat{A} \in \mathcal{A}} \hat{A} = \beta(\mathbb{N})$. We also have

$$\mathcal{U} \in \hat{A} \cap \hat{B} \Leftrightarrow A, B \in \mathcal{U} \Leftrightarrow A \cap B \in \mathcal{U} \Leftrightarrow \mathcal{U} \in \widehat{A \cap B}$$

Thus the set of all unions of the members of $\mathcal{A}$ defines a topology. □

Similarly, one can show that $\mathcal{A}$ is closed under complements, and thus every set $\hat{A}$ is both closed and open.

Proposition 2.29. *$\beta(\mathbb{N})$ is compact, Hausdorff, and the principal ultrafilters are dense in $\beta(\mathbb{N})$.*

As a corollary of this, we can conclude that $C(\beta(\mathbb{N}))$, the space of continuous functions on $\beta(\mathbb{N})$, is a C^*-algebra with conjugation as the $*$ operator.

Furthermore, one can show that $\mathbb{D}^2$ and $C(\beta(\mathbb{N}))$ are isometrically isomorphic, extending $f \in \mathbb{D}^2$ uniquely to $f \in C(\beta(\mathbb{N}))$ by $f(\mathcal{U}) = \mathcal{U}\text{-lim} f_n$, where we define $\mathcal{U}$-lim as follows.
Given a bounded sequence $(a_1, a_2, \ldots), a_i \in \mathbb{C}$, this sequence may not have a limit in the normal sense. However, given an ultrafilter $\mathcal{U}$, we can define another limit point, that is unique if it exists.

Definition 2.30. Given a bounded sequence $(a_1, a_2, \ldots)$, and an ultrafilter $\mathcal{U} \in \beta(\mathbb{N})$, we say a point $x \in \mathbb{C}$ is a $\mathcal{U}$-limit of a if, for every neighbourhood S of x, we have $\{i : a_i \in S\} \in \mathcal{U}$. We say

$$x = \mathcal{U}\text{-lim}(a_i).$$

It follows quite easily from definition that:

- $\mathcal{U}\text{-lim}(a_i + b_i) = \mathcal{U}\text{-lim}(a_i) + \mathcal{U}\text{-lim}(b_i)$
- $\mathcal{U}\text{-lim}(\lambda a_i) = \lambda \mathcal{U}\text{-lim}(a_i)$

Thus $\mathbb{D}^2$ is a C^*-algebra. This isometry implies a stronger connection between $\mathbb{D}^2$ and $\beta(\mathbb{N})$, which manifests as follows.

Definition 2.31. For any ultrafilter $\mathcal{U} \in \beta(\mathbb{N})$, we define the linear functional $f_{\mathcal{U}} : \mathbb{D}^2 \to \mathbb{C}$ by

$$f_{\mathcal{U}}(D) := \mathcal{U}\text{-lim}(\text{diag} D)$$

where $\text{diag} D = (\langle De_1, e_1\rangle, \langle De_2, e_2\rangle, \ldots)$.

Theorem 2.32. *The pure states on $\mathbb{D}^2$ are exactly the functions $\{f_{\mathcal{U}} : \mathcal{U} \in \beta(\mathbb{N})\}$.*

Proof. See the introduction of [33]. We shall sketch a proof here for the interested reader. $\mathbb{D}^2 \simeq C(\beta(\mathbb{N}))$ and so the dual space, by the Riesz-Markov-Kakutani theorem theorem[22, IV.6.3], can be identified with the space of Borel measures on $\beta(\mathbb{N})$. States then correspond to regular Borel probability measures of $\beta(\mathbb{N})$, [31, Chp. 2]. Pure states correspond to extreme points, which are precisely the point mass measures and are in one to one correspondence with ultrafilters of $\mathbb{N}$. $\square$

Using this result, we could have quickly found all pure states on

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},\ a, d \in \mathbb{C} \right\}$$

by noting that the only ultrafilters on $\{1, 2\}$ are the principal ultrafilters

$$\mathcal{U}_1 = \{\{1\}, \{1, 2\}\}$$
$$\mathcal{U}_2 = \{\{2\}, \{1, 2\}\}$$

which give rise to the pure states $f_1(a, d) = a$ and $f_2(a, d) = d$.

2.4 Statement of the Kadison-Singer Problem

Theorem 2.33. *Let $\mathcal{A}$ be a discrete MASA of $\mathcal{B}(\mathcal{H})$, the algebra of bounded linear operators on a separable Hilbert space. Let $\rho : \mathcal{A} \to \mathbb{C}$ be a pure state on that subalgebra. Then there exists a pure state extension $\rho' : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ of ρ that is unique.*

For purposes of this discussion, we shall take $\mathcal{H} = \ell_2$ and $\mathcal{A}$ to be the space of bounded diagonal operators; the ideas presented below transfer to the more general case, but notation and theory are less dense.

We can easily show the existence of a pure state extension, as in [25, Theorem 4.3.13], or [35].

Theorem 2.34 (Krein-Milman)**.** *If S is convex and compact in a locally convex space, then S is the closed convex hull of its extreme points. In particular, S has extreme points.*

Proposition 2.35. *Let $\rho : \mathbb{D}^2 \to \mathbb{C}$ be a pure state on the space of bounded diagonal operators. Then there exists a pure state extension $\rho' : \mathcal{B}(\ell_2) \to \mathbb{C}$ of ρ.*

Proof. We first define, for any $H \in \mathcal{B}(\ell_2)$, the *diagonal* of H to be $D(H)$ where

$$\langle D(H)e_i, e_j \rangle = \begin{cases} \langle He_i, e_j \rangle & i = j \\ 0 & i \neq j \end{cases}$$

Then, for a pure state f on $\mathbb{D}^2$, define $g : \mathcal{B}(\ell_2) \to \mathbb{C}$ by $g(H) := f(D(H))$

g is clearly linear and $g(\mathbb{I}) = f(D(\mathbb{I}) =)f(\mathbb{I}) = 1$. Now suppose $H \in \ell_2$ is positive. Then $\langle He_i, e_i \rangle \in \{x \in \mathbb{R} : x \geq 0\}$ for all i and so $D(H)$ is clearly positive. Thus $g(H) = f(D(H)) \in \{x \in \mathbb{R} : x \geq 0\}$. Thus g is a state extension.

Then note that the set of extensions of f to a state of $\mathcal{B}(\ell_2)$ is weak*-compact and convex. Thus, if we start this a pure state f, by the Krein-Milman theorem, the set of extensions has an extreme point. By convexity of the set, and the definition of extreme points, this extreme point is a pure state of $\mathcal{B}(H)$. □

However, proof of its uniqueness is quite challenging, as so we shall instead discuss the proof of an equivalent statement.

3 Equivalent Formulations

3.1 Anderson's Paving Conjecture

Theorem 3.1. *For every $\epsilon > 0$, there exists $r \in \mathbb{N}$ such that, for every self adjoint $H \in \mathcal{B}(\ell_2)$ with $D(H) = 0$, there exists a partition $\{A_1, A_2, \ldots, A_r\}$ of $\mathbb{N}$ such that $\|P_{A_i} H P_{A_i}\| \leq \epsilon \|H\|$ for all $1 \leq i \leq r$.*

This is one of the first and most commonly discussed equivalent formulation, called Anderson's Infinite Dimensional Paving Conjecture, introduced in [3][2][4]. It can in fact, be simplified further to a finite dimensional case. The above statement can be shown to be equivalent to the following.

Claim 3.2. *For every $\epsilon > 0$, there exists $r \in \mathbb{N}$ such that, for every self adjoint $H \in \mathcal{B}({\ell_2}^n)$ with $D(H) = 0$, there exists a partition $\{A_1, A_2, \ldots, A_r\}$ of $\{1, 2, \ldots, n\}$ such that $\|P_{A_i} H P_{A_i}\| \leq \epsilon \|H\|$ for all $1 \leq i \leq r$. Here ${\ell_2}^n$ is $\mathbb{C}^n$ with the ℓ_2 norm.*

For proof of this, we recommend [33] or [10]. We will only discuss the equivalence of the Kadison-Singer Problem and Anderson's Infinite-Dimensional Paving Conjecture.

We state the following without proof.

Lemma 3.3. *Let $\mathcal{U} \in \beta(\mathbb{N})$ and let $f_{\mathcal{U}} : \mathbb{D}^2 \to \mathbb{C}$ be the corresponding pure state. Then the following are equivalent*

1. *$f_{\mathcal{U}}$ has a unique extension to a state on $\mathcal{B}(\ell_2)$.*
2. *For every self-adjoint $H \in \mathcal{B}(\ell_2)$ with $D(H) = 0$ and every $\epsilon > 0$, there exists $A \in \mathcal{U}$ with $-\epsilon P_A \preceq P_A H P_A \preceq \epsilon P_A$.*

Theorem 3.4. *The following are equivalent:*

1. *Anderson's Infinite Dimensional Paving Conjecture holds*
2. *For every $\epsilon > 0$ and every self-adjoint $H \in \mathcal{B}(\ell_2)$ with $D(H) = 0$, there exists $r \in \mathbb{N}$ and a partition $A_1, \ldots, A_r$ or $\mathbb{N}$ such that $-\epsilon P_{A_j} \preceq P_{A_j} H P_{A_j} \preceq \epsilon P_{A_j}$ for every $1 \leq j \leq r$.*
3. *The Kadison-Singer Problem holds.*

Proof. We now show that (1) and (3) are both equivalent to (2).

(1) $\Rightarrow$ (2) It is straight forward to show that $\|P_A H P_A\| \leq \epsilon \|H\|$ implies $-\epsilon P_A \preceq P_A H P_A \preceq \epsilon P_A$ for any $A \subset \mathbb{N}$.

(2) $\Rightarrow$ (1) Suppose (2) holds, but (3) does not. Then there exists and $\epsilon > 0$ such that, for every $r \in \mathbb{N}$, there exists a self-adjoint $H_r \in \mathcal{B}(\ell_2)$ with $D(H_r) = 0$, but no partition $A_1, \ldots, A_r$ of $\mathbb{N}$ satisfies $\|P_{A_i} H_r P_{A_i}\| \leq \epsilon \|H_r\|$ for all $1 \leq i \leq r$. This implies, that no partition exists so that $-\epsilon P_{A_i} \preceq P_{A_i} H_r P_{A_i} \preceq \epsilon P_{A_i}$ for every $1 \leq i \leq r$.

Without loss of generality, we may assume $\|H_r\| = 1$. Define $H = \bigoplus_{i=1}^{\infty} H_i$. As the countable product of countable sets is countable, H is an operator on ℓ_2. We have that H is self-adjoint, $D(H) = 0$ and that $\|H\| = 1$, so $H \in \mathcal{B}(\ell_2)$. Then, by (2), there exists $r \in \mathbb{N}$ and a partition $A_1, \ldots, A_r$ of $\mathbb{N}$ such that $-\epsilon P_{A_i} \preceq P_{A_i} H P_{A_i} \preceq \epsilon P_{A_i}$ for every $1 \leq i \leq r$. Restricting to the r^{th} part of the direct sum, we get $-\epsilon P_{A_i} \preceq P_{A_i} H_r P_{A_i} \preceq \epsilon P_{A_i}$ for every $1 \leq i \leq r$, a contradiction.

(2) $\Rightarrow$ (3) Choose an ultrafilter $\mathcal{U} \in \beta(\mathbb{N})$, a self-adjoint $H \in \mathcal{B}(\ell_2)$ with $D(H) = 0$, and an $\epsilon > 0$. By (2), there exists a partition $A_1, \ldots, A_r$ of $\mathbb{N}$ such that $-\epsilon P_{A_i} \preceq P_{A_i} H P_{A_i} \preceq \epsilon P_{A_i}$ for every $1 \leq i \leq r$. By Lemma 2.26, some $A_i \in \mathcal{U}$. Since this is true for all such H and ϵ, Lemma 3.3 implies $f_{\mathcal{U}}$ extends uniquely to $\mathcal{B}(\ell_2)$. As all pure states are of the form $f_{\mathcal{U}}$ for some $\mathcal{U} \in \beta(\mathbb{N})$, by Theorem 2.32, and $\mathcal{U}$ was arbitrary, this implies (3).

(3) $\Rightarrow$ (2) Choose a self-adjoint $H \in \mathcal{B}(\ell_2)$ with $D(H) = 0$ and choose $\epsilon > 0$. Since (3) holds, Lemma 3.3 tells us that, for every $\mathcal{U} \in \beta(\mathbb{N})$, there exists $A_{\mathcal{U}} \in \mathcal{U}$ with $-\epsilon P_{A_{\mathcal{U}}} \preceq P_{A_{\mathcal{U}}} H P_{A_{\mathcal{U}}} \preceq \epsilon P_{A_{\mathcal{U}}}$.

Then $\{\hat{A}_{\mathcal{U}} : \mathcal{U} \in \beta(\mathbb{N})\}$ is an open cover of $\beta(\mathbb{N})$. As $\beta(\mathbb{N})$ is compact, by Theorem 2.29, there is a finite subcover $\{\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_k\}$ such that $-\epsilon P_{A_i} \preceq P_{A_i} H P_{A_i} \preceq \epsilon P_{A_i}$ for all $1 \leq i \leq r$. Then note that $A_1 \cup \cdots \cup A_k = \mathbb{N}$, if $\hat{A}_1 \cup \cdots \cup \hat{A}_k = \beta(\mathbb{N})$. While this may not form a partition of $\mathbb{N}$, the partition of $\mathbb{N}$ obtained by all possible intersections of $A_1, \ldots, A_k$ will give the desired result with $r \leq 2^k$.

□

From this, and the reduction to Anderson's Finite Dimensional Paving Conjecture, we can establish many further equivalent formulations. See [30], [10] and [1] for further details

Theorem 3.5. *The following are equivalent:*

1. *The Kadison-Singer Problem holds.*

2. $\forall \epsilon > 0$, $\exists r \in \mathbb{N}$, $\forall n \in \mathbb{N}$ *and for every self-adjoint matrix* $A \in \mathbb{C}^{n \times n}$ *with* $D(A) = 0$, *there exists a partition* $A_1, \ldots, A_r$ *of* $\{1, 2, \ldots, n\}$ *such that* $\|P_{A_i} A P_{A_i}\| \leq \epsilon \|A\|$ *for all* $1 \leq i \leq r$.

3. $\forall \epsilon > 0$, $\exists r \in \mathbb{N}$, $\forall n \in \mathbb{N}$ *and for every matrix* $A \in \mathbb{C}^{n \times n}$ *with* $D(A) = 0$, *there exists a partition* $A_1, \ldots, A_r$ *of* $\{1, 2, \ldots, n\}$ *such that* $\|P_{A_i} A P_{A_i}\| \leq \epsilon \|A\|$ *for all* $1 \leq i \leq r$.

4. $\forall \epsilon > 0$, $\exists r \in \mathbb{N}$, $\forall n \in \mathbb{N}$ *and for every matrix* $R \in \mathbb{C}^{n \times n}$ *with* $R = R^* = R^{-1}$ *and* $D(R) = 0$, *there exists a partition* $A_1, \ldots, A_r$ *of* $\{1, 2, \ldots, n\}$ *such that* $\|P_{A_i} R P_{A_i}\| \leq \epsilon \|A\|$ *for all* $1 \leq i \leq r$.

5. $\forall \epsilon > 0$, $\exists r \in \mathbb{N}$, $\forall n \in \mathbb{N}$ *and for every matrix* $Q \in \mathbb{C}^{n \times n}$ *with* $Q = Q^* = Q^2$ *and* $D(Q) = \frac{1}{2}\mathbb{I}$, *there exists a partition* $A_1, \ldots, A_r$ *of* $\{1, 2, \ldots, n\}$ *such that* $\|P_{A_i} Q P_{A_i}\| \leq \epsilon \|A\|$ *for all* $1 \leq i \leq r$.

3.2 The Feichtinger Conjecture

Theorem 3.6. *Every bounded frame is a finite union of Riesz basic sequences.*

Definition 3.7. A family of vectors $\{v_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ is a ***Riesz basic sequence*** if there exist $A, B > 0$ such that, for all scalar sequences $\{a_i\}_{i\in I}$ we have:

$$A\sum_{i\in I}|a_i|^2 \leq \|\sum_{i\in I}a_iv_i\|^2 \leq B\sum_{i\in I}|a_i|^2$$

We call $\sqrt{A}, \sqrt{B}$, ***lower and upper Riesz basis bounds*** for $\{v_i\}_{i\in I}$. If$A = B$, we call this a B***-tight frame***. If a Riesz basic sequence $\{v_i\}_{i\in I}$ spans $\mathcal{H}$, we call it a ***Riesz basis*** for $\mathcal{H}$.

Hilbert frames were introduced in [21] to analyse problems in non harmonic Fourier series. We define a frame as follows.

Definition 3.8. A family $\{v_i\}_{i\in I}$ of vectors in a Hilbert space $\mathcal{H}$ is called a ***frame*** if there exist constants $0 < A \leq B < \infty$, called the ***lower and upper frame bounds***, so that for all $u \in \mathcal{H}$

$$A\|u\|^2 \leq \sum_{i\in I}|\langle u, v_i\rangle|^2 \leq B\|u\|^2$$

If we can find such a B, but no such A exists, we call $\{v_i\}_{i\in I}$ a ***Bessel sequence*** with Bessel bound B.

In [12], it is shown that the Feichtinger Conjecture is equivalent to the following.

Claim 3.9. *Every bounded Bessel sequence can be written as a finite union of Riesz basic sequences. Equivalently, for every $B > 0$, there exists $n_B \in \mathbb{N}$ and $A_B \geq 0$ so that every Bessel sequence $\{v_i\}_{i=1}^n$ with Bessel bound B and $\|v_i\| = 1$ for all $1 \leq i \leq n$ can be written as a union of n_B Riesz basic sequences with lower Riesz basis bound A_B.*

Theorem 3.10. *The following are equivalent:*

1. *The Kadison-Singer Problem holds.*
2. *For every $B > 0$, there exists $n_B \in \mathbb{N}$ and $A_B \geq 0$ so that every Bessel sequence $\{v_i\}_{i=1}^n$ with Bessel bound B and $\|v_i\| = 1$ for all $1 \leq i \leq n$ can be written as a union of n_B Riesz basic sequences with lower Riesz basis bound A_B.*

Proof. See [17][14], in which (2) is shown to be equivalent to the Paving Conjecture. As the proof is highly involved, we will not provide the details here. ☐

3.3 The R_ϵ Conjecture

Theorem 3.11. *For every $\epsilon > 0$, every unit norm Riesz basic sequence is a finite union of ϵ-Riesz basic sequences.*

Definition 3.12. Let $\{v_i\}_{i\in I}$ be a Riesz basic sequence. If $\epsilon > 0$ is such that

$$(1-\epsilon)\sum_{i\in I}|a_i|^2 \le \|\sum_{i\in I} a_i v_i\|^2 \le (1+\epsilon)\sum_{i\in I}|a_i|^2$$

for any scalar sequence $\{a_i\}_{i\in I}$, we call $\{v_i\}_{i\in I}$ a ***ϵ-Riesz basic sequence***. If $\|v_i\| = 1$ for all $i \in I$, we call it a ***unit norm*** Riesz basic sequence.

The R_ϵ-Conjecture was first stated in [14], but it was shown in [11] that it was equivalent to the Kadison-Singer Problem. We shall outline the proof here.

Theorem 3.13. *The following are equivalent:*

1. *The Paving Conjecture holds.*
2. *If $T : \ell_2 \to \ell_2$ is a bounded linear operator with $\|Te_i\| = 1$ $\forall i \in I$, then for every $\epsilon > 0$, $\{Te_i\}_{i\in I}$ is a finite union of ϵ-Riesz basic sequences.*
3. *The R_ϵ-Conjecture holds.*

Proof. We shall show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) Choose $\epsilon > 0$. Given T as in (2), define $S = T^*T$. Since $D(S) = \mathbb{I}$, the Paving conjecture implies the existence of $r \in \mathbb{N}$ and a partition $A_1, \ldots, A_r$ of I such that, for every $1 \le i \le r$

$$\|P_{A_i}(\mathbb{I} - S)P_{A_i}\| \le \delta\|\mathbb{I} - S\|$$

where $\delta = \frac{\epsilon}{\|S\|+1}$. Then we can show, for all $u = \sum_{i\in I} a_i e_i$, that

$$\begin{aligned}
\|\sum_{i\in A_j} a_i Te_i\|^2 &= \|TP_{A_j}u\|^2 \\
&= \langle TP_{A_j}u, TP_{A_j}u\rangle \\
&= \langle T^*TP_{A_j}u, P_{A_j}u\rangle \\
&= \langle P_{A_j}u, P_{A_j}u\rangle - \langle P_{A_j}(\mathbb{I} - S)P_{A_j}u, P_{A_j}\rangle \\
&\ge \|P_{A_j}u\|^2 - \delta\|\mathbb{I} - S\|\|P_{A_j}u\|^2 \\
&\ge (1-\epsilon)\|P_{A_j}u\|^2 = (1-\epsilon)\sum_{i\in A_j}|a_i|^2.
\end{aligned}$$

Similarly $\|\sum_{i\in A_j} a_i Te_i\|^2 \le (1+\epsilon)\sum_{i\in A_j}|a_i|^2$.

(2) $\Rightarrow$ (3) This is reasonably trivial.

(3) $\Rightarrow$ (1) Choose $T \in \mathcal{B}(\ell_2)$ be such that $\|Te_i\| = 1$ for all $i \in I$ and let $v_i = Te_i$. It is know that that the Paving Conjecture is true if it holds for *Gram* operators, [5], [6], [7], [23]. So we shall show that the Paving Conjecture holds for the Gram operator of T G. We define G by $\langle Ge_i, e_j\rangle = \langle v_j, v_i\rangle$ for all $i, j \in I$.

Fix $0 < \delta < 1$ and let $\epsilon > 0$. Define $g_i = \sqrt{1-\delta^2} v_i \oplus \delta e_i \in \ell_2 \oplus \ell_2$. Then $\|g_i\| = 1$ and for all scalars $\{a_i\}_{i\in I}$

$$\begin{aligned}\delta \sum_{i\in I} |a_i|^2 \le \| \sum_{i\in I} a_i g_i\|^2 &= (1-\delta^2)\| \sum_{i\in I} a_i Te_i\|^2 + \delta^2 \sum_{i\in I} |a_i|^2 \\ &\le ((1-\delta^2)\|T\|^2 + \delta^2)) \sum_{i\in I} |a_i|^2.\end{aligned}$$

Thus $\{g_i\}_{i\in I}$ is a unit norm Riesz basic sequence and $\langle g_i, g_j\rangle = (1-\delta^2)\langle v_i, v_j\rangle$, for all $i \neq j$. Thus, by the R_ϵ-Conjecture, there is a partition, $\{A_k\}_{k=1}^r$ so that, for all $1 \le k \le r$ and all $f = \sum_{i\in I} a_i e_i$

$$\begin{aligned}(1-\epsilon) \sum_{i\in A_k} |a_i^2 \le \| \sum_{i\in A_k} a_j g_i\|^2 &= \langle \sum_{i\in A_k} a_j g_i\rangle \\ &= \sum_{i\in A_k} |a_i|^2 \|g_i\|^2 + \sum_{i\neq j\in A_k} a_i \bar{a_j} \langle g_i, g_j\rangle \\ &= \sum_{i\in A_k} |a_i|^2 + (1-\delta^2) \sum_{i\neq j\in A_k} a_i \bar{a_j} \langle f_i, f_j\rangle \\ &= \sum_{i\in A_k} |a_i|^2 + (1-\delta^2)\langle P_{A_k}(G - D(G))P_{A_k} f, f\rangle \\ &\le (1+\epsilon) \sum_{i\in A_k} |a_i|^2.\end{aligned}$$

This implies

$$-\epsilon \sum_{i\in A_k} |a_i|^2 \le (1-\delta^2)\langle P_{A_k}(G - D(G))P_{A_k} f, f\rangle \le \epsilon \sum_{i\in A_k} |a_i|^2.$$

That is to say

$$(1-\delta^2)\|P_{A_k}(G - D(G))P_{A_k}\| \le \epsilon$$

as $P_{A_k}(G - D(G))P_{A_k}$ is self-adjoint.

$\square$

3.4 The Bourgain-Tzafriri Conjecture

Theorem 3.14. *There are universal constants $A > 0$ so that for every $B > 1$ there exists $r_B \in \mathbb{N}$ such that, whenever $T : {\ell_2}^n \to {\ell_2}^n$ is a linear operator with $\|Te_i\| = 1$ for $1 \leq i \leq n$, then there exists a partition $A_1, \ldots A_{r_B}$ of $\{1, 2, \ldots, n\}$ so that, for all $j = 1, 2, \ldots, n$ and all choices of scalars $\{a_j\}_{j \in S}$, we have*

$$\| \sum_{i \in A_j} a_i T e_i \|^2 \geq A \sum_{i \in A_j} |a_i|^2.$$

This conjecture arose from a theorem of Bourgain and Tzafriri in a paper from 1987 [8], called the ***restricted invertibility principle***. It received a great deal of attention, [9], [11], [14] leading to a proof in [14] that the Bourgain-Tzafriri Conjecture and the Kadison-Singer Problem were equivalent.

Theorem 3.15 (Restricted Invertibility)**.** *There exist universal constants $A, c > 0$, so that, whenever $T : {\ell_2}^n \to {\ell_2}^n$ is a a linear operator with $\|Te_i\| = 1$ for $1 \leq i \leq n$, then there exists a subset $S \subset \{1, 2, \ldots, n\}$ of cardinality $|S| \geq \frac{cn}{\|T\|^2}$ so that, for all $1 \leq j \leq n$ and all choice of scalars $\{a_j\}_{j \in S}$*

$$\| \sum_{j \in S} a_j T e_j \|^2 \geq A \sum_{j \in S} |a_j|^2,$$

The above conjecture was often called the ***strong*** form of the conjecture, as there exists a weakening of it, in which the constant A is a function of $\|T\|$. Significant effort was made to show that the strong and weak forms of the conjecture were equivalent, [5][14]. In [11], success was found, by showing that both forms were equivalent to the following.

Conjecture 3.16. There exists a constant $A > 0$ and $r \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, if $T : {\ell_2}^n \to {\ell_2}^n$ has $\|Te_i\| = 1$ for all $1 \leq i \leq n$ and $\|T\| \leq 2$, then there exists a partition $A_1, \ldots, A_r$ of $\{1, 2, \ldots, n\}$ so that, for all $1 \leq j \leq r$ and all scalars $\{a_i\}_{i \in A_j}$ we have

$$\| \sum_{i \in A_j} a_i T e_i \|^2 \geq A \sum_{i \in A_j} |a_i|^2.$$

From this result, we can prove the following.

Theorem 3.17. *The following are equivalent:*

1. *The Kadison-Singer Problem.*
2. *The strong Bourgain-Tzafriri Conjecture.*
3. *The weak Bourgain-Tzafriri Conjecture.*
4. *Conjecture 3.16.*

Proof. We shall show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) The truth of the Kadison-Singer Problem implies the truth of the R_ϵ Conjecture, which immediately implies the strong Bourgain-Tzafriri Conjecture.

(2) $\Rightarrow$ (3) If there exists a universal constant A, then choosing $A(\|T\|) = A$ gives the necessary function for the weak conjecture.

(3) $\Rightarrow$ (4) If there exists a function $A(\|T\|) > 0$, we choose $A = \inf_{\|T\|}\{A(\|T\|)\}$, which will be non-zero as $[0, 2]$ is compact.

(4) $\Rightarrow$ (1) It here suffices to show (4) implies the following conjecture, due to a result by Akemann and Anderson in [1], in which they show that this conjecture implies the truth of the Kadison-Singer Problem.

Conjecture 3.18. There exist universal constants $0 < \delta, \epsilon < 1$ and $r \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ and all orthogonal projections P on ${\ell_2}^n$ with $\max_{1\leq i\leq n} |p_{ii}| \leq \delta$, there is a partition $A_1, A_2, \ldots, A_r$ of $\{1, \ldots, n\}$ so that $\|P_{A_j} P P_{A_j}\| \leq 1 - \epsilon$, for all $1 \leq j \leq r$.

So let A, r satisfy Conjecture 3.16 and fix $0 < \delta \leq \frac{3}{4}$. Let P be an orthogonal projection as in Conjecture 3.18. Now, $\langle Pe_i, e_i\rangle = \|Pe_i\|^2 \leq \delta$, and so $\|(\mathbb{I} - P)e_i\|^2 \geq 1 - \delta \geq \frac{1}{4}$. Now define $T : {\ell_2}^n \to {\ell_2}^n$ by

$$Te_i = \frac{(\mathbb{I} - P)e_i}{\|(\mathbb{I} - P)e_i\|}/$$

For any scalars $\{a_i\}_{i=1}^n$, we have

$$\begin{aligned}
\|\sum_{i=1}^n a_i Te_i\|^2 &= \|\sum_{i=1}^n \frac{a_i}{\|(\mathbb{I} - P)e_i\|}(\mathbb{I} - P)e_i\|^2 \\
&\leq \sum_{i=1}^n \left|\frac{a_i}{\|(\mathbb{I} - P)e_i\|}\right|^2 \\
&\leq 4\sum_{i=1}^n |a_i|^2.
\end{aligned}$$

Then note $\|Te_i\| = 1$ and $\|T\| \leq 2$, and, thus, there is a partition $A_1, \ldots, A_r$ of $\{1, \ldots, n\}$ such that, for all $1 \leq j \leq r$ and all scalars $\{a_i\}_{i\in A_j}$, we have

$$\|\sum_{i\in A_j} a_i T_e i\|^2 \geq A \sum_{i\in A_j} |a_i|^2.$$

Hence

$$\|\sum_{i\in A_j} a_i(\mathbb{I}-P)e_i\|^2 = \|\sum_{i\in A_j} a_i\|(\mathbb{I}-P)e_i\|Te_i\|^2$$
$$\geq A\sum_{i\in A_j}|a_i|^2\|(\mathbb{I}-P)e_i\|^2$$
$$\geq \frac{A}{4}\sum_{i\in A_j}|a_i|^2.$$

So for any scalars $\{a_i\}_{i\in A_j}$, we have

$$\sum_{i\in A_j}|a_i|^2 = \|\sum_{i\in A_j} a_iPe_i\|^2 + \|\sum_{i\in A_j} a_i(\mathbb{I}-P)e_i\|^2$$
$$\geq \|\sum_{i\in A_j} a_iPe_i\|^2 + \frac{A}{4}\sum_{i\in A_j}|a_i|^2.$$

Thus, for any $u=\sum_{i=1}^n a_ie_i$, we have

$$\|PP_{A_j}u\|^2 = \|\sum_{i\in A_j} a_iPe_i\|^2 \leq (1-\frac{A}{4})\sum_{i\in A_j}|a_i|^2$$

and thus

$$\|P_{A_j}PP_{A_j}\| = \|PP_{A_j}\|^2 \leq 1-\frac{A}{4}.$$

So Conjecture 3.18 holds.

□

Note that combining the proof of Theorem 3.13 and the above theorem implies that the Kadison-Singer Problem is in fact equivalent to a weaker version of the R_ϵ-Conjecture. We do not require the upper inequality, nor that the lower constant be close to one.

3.5 Weaver's KS_r

Theorem 3.19. *There exist universal constants $N \geq 2$ and $\epsilon > 0$ such that the following holds. Let $v_1, v_2, \ldots, v_n \in \mathbb{C}^k$ satisfy $\|v_i\| \leq 1$ for all $1 \leq i \leq n$ and suppose*

$$\sum_{i=1}^{n} |\langle u, v_i \rangle|^2 \leq N$$

for every unit vector $u \in \mathbb{C}^k$. Then there exists a partition $A_1, A_2, \ldots A_r$ of $\{1, 2, \ldots, n\}$ such that

$$\sum_{i \in A_j} |\langle u, v_i \rangle|^2 \leq N - \epsilon$$

for every unit vector $u \in \mathbb{C}^k$ and all $1 \leq j \leq r$.

Note that N and ϵ must be independent of both n and k. Weaver proves that KS_r implies the Kadison-Singer Problem much like our proof of Theorem 3.17, by showing that KS_r implies Conjecture 3.18. As the proof of the reverse implication is quite involved, we will simply refer the reader to Weaver's proof in [34].

Theorem 3.20. *The following are equivalent:*

1. *KS_r holds for some $r \in \mathbb{N}$.*
2. *The Kadison-Singer Problem holds.*

(1) $\Rightarrow$ (2). Suppose KS_r holds for some r, N, ϵ. Let P be an orthogonal projection with $\max_{1 \leq i \leq n} |p_{ii}| \leq \frac{1}{N}$. If P has rank k, then its range is a subspace $V \subset \mathbb{C}^n$ of dimension k. Define $v_i = \sqrt{N} P e_i \in V$ for $1 \leq i \leq n$ and note that

$$\|v_i\|^2 = N \|Pe_i\|^2 = N \langle Pe_i, e_i \rangle \leq N \cdot \frac{1}{N} = 1$$

for all $1 \leq i \leq n$. Note also that, for any unit vector $u \in V$,we have

$$\sum_{i=1}^{n} |\langle u, v_i \rangle^2 = \sum_{i=1}^{n} |\langle u, \sqrt{N} Pe_i \rangle|^2 = N \sum_{i=1}^{n} |\langle u, e_i \rangle|^2 = N.$$

Thus KS_r gives the existence of a partition $A_1, \ldots, A_r$ of $\{1, \ldots, n\}$ such that

$$\sum_{i \in A_j} |\langle u, v_i \rangle|^2 \leq N - \epsilon$$

for every unit vector $u \in V$ and every $1 \leq j \leq r$. We then have, for any unit vector $u \in V$ that

$$\begin{aligned}\|P_{A_j}Pu\|^2 &= \sum_{i=1}^{n} |\langle P_{A_j}Pu, e_i\rangle|^2 = \sum_{i=1}^{n} |\langle u, PP_{A_j}e_i\rangle|^2 \\ &= \frac{1}{N}\sum_{i\in A_j} |\langle u, v_i\rangle|^2 \leq 1 - \frac{\epsilon}{N}\end{aligned}$$

and thus $\|P_{A_j}PP_{A_j}\| = \|P_{A_j}P\|^2 \leq 1 - \frac{\epsilon}{N}$, as required. □

Weaver also provides many many modifications of the KS_r conjecture that remain equivalent to the Kadison-Singer Problem.

Theorem 3.21. *KS_r remains equivalent to the Kadison-Singer Problem under either or both of the following modifications:*

1. *Require $\epsilon = 1$.*
2. *Assume $\sum_{i=1}^{n} |\langle u, v_i\rangle|^2 = N$ for every unit vector u instead of $\sum_{i=1}^{n} |\langle u, v_i\rangle|^2 \leq N$.*

Another equivalent formulation of KS_r has that v_i have unit length, but the freedom of choice of ϵ is massively reduced. This conjecture is referred to as KS'_r.

Conjecture 3.22. There exist universal constants $N \geq 4$ and $\epsilon > \sqrt{N}$ such that the following holds. Let $v_1, \dots, v_n \in \mathbb{C}^k$ satisfy $\|v_i\| = 1$ and suppose

$$\sum_{i=1}^{n} |\langle u, v_i\rangle|^2 \leq N$$

for every unit vector $u \in \mathbb{C}^k$. Then there exists a partition $A_1, \dots, A_r$ of $\{1, \dots, n\}$ such that

$$\sum_{i\in A_j} |\langle u, v_i\rangle|^2 \leq N - \epsilon$$

for every unit vector $u \in \mathbb{C}^k$ and all $1 \leq j \leq r$.

Weaver also proved some partial results regarding KS_2, which is the case proven by Marcus, Spielman and Srivastava in 2014 [28]. KS_2 can be restated as follows.

Theorem 3.23. *There exist universal constants $N \geq 2$ and $\epsilon > 0$ such that the following holds. Let $v_1, v_2, \dots, v_n \in \mathbb{C}^k$ satisfy $\|v_i\| \leq 1$ for all $1 \leq i \leq n$ and suppose*

$$\sum_{i=1}^{n} |\langle u, v_i\rangle|^2 \leq N$$

for every unit vector $u \in \mathbb{C}^k$. Then there some choice of signs such that

$$\sum_{i=1}^{n} \pm|\langle u, v_i\rangle|^2 \leq N - \epsilon$$

for every unit vector $u \in \mathbb{C}^k$.

4 Proving KS_r

KS_2 was shown in 2014 by Adam Marcus, Daniel Spielman and Nikhil Srivastava [28], using techniques from linear algebra and random matrix theory. In particular, they introduced new techniques involving the following, notoriously troublesome, representation of the operator norm

$$\|A\|_{op} = \text{maxroot}(p_A)$$

where p_A is the characteristic polynomial of A and $\text{maxroot}(p)$ is the largest real root of a non-zero polynomial p. Using these techniques to analyse certain families of random matrices and interlacing families of their characteristic polynomials to prove the following theorem.

Theorem 4.1. *If $\epsilon > 0$ and $v_1, v_2, \ldots, v_n$ are independent random vectors in $\mathbb{C}^k$ with finite support such that*

$$\sum_{i=1}^{n} \mathbb{E} v_i v_i^* = \mathbb{I}$$

and

$$\mathbb{E}\|v_i\|^2 \leq \epsilon \text{ for all } 1 \leq i \leq n$$

then

$$\mathbb{P}\left[\left\|\sum_{i=1}^{n} v_i v_i^*\right\| \leq (1+\sqrt{\epsilon})^2\right] > 0.$$

We use the following definitions.

Definition 4.2. Let X be a random variable taking values in $\mathcal{X}$, with probability density function f_X and let $\mathcal{S} \subset \mathcal{X}$ be a subset. We denote the ***probability*** of x taking a value in $\mathcal{S}$ by

$$\mathbb{P}[X \in \mathcal{S}] := \int_{\mathcal{S}} f_X d\mu$$

where μ is the probability measure on $\mathcal{X}$.

We define the ***mean***, or ***first moment***, similarly

$$\mathbb{E}X := \int_{\mathcal{S}} X f_X d\mu.$$

Theorem 4.1 has the following generalization of KS_2 as a corollary

Corollary 4.3. *Let $r \in \mathbb{N}$ and let $u_1, \ldots, u_n \in \mathbb{C}^k$ be vectors such that*

$$\sum_{i=1}^{n} u_i u_i * = \mathbb{I}$$

and $\|u_i\|^2 \leq \delta$ *for all* $1 \leq i \leq n$*. Then there exists a partition* $A_1, \ldots, A_r$ *of* $\{1, \ldots, n\}$ *such that*

$$\left\| \sum_{i \in A_j} u_i u_i^* \right\| \leq (\frac{1}{\sqrt{r}} + \sqrt{\delta})^2$$

for $1 \leq j \leq r$*.*

Proof. For each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, r\}$, define $w_{i,j} \in \mathbb{C}^{rk}$ to be the direct sum of r vectors from $\mathbb{C}^k$, all of which are the zero vector expect the j^{th}, which we take to be u_i.
Now let $v_1, v_2, \ldots, v_n$ to be independent random vectors such that v_i takes the values $\{\sqrt{r}w_{i,j}\}_{j=1}^r$ each with equal probability $\frac{1}{r}$. These vectors then satisfy $\|v_i\|^2 = r\|u_i\|^2 \leq r\delta$ for each $1 \leq i \leq n$. We also have

$$\mathbb{E}v_i v_i* = \begin{pmatrix} u_i u_i^* & 0 & \cdots & 0 \\ 0 & u_i u_i* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_i u_i^* \end{pmatrix}$$

And so

$$\sum_{i=1}^n \mathbb{E}v_i v_i^* = \mathbb{I}.$$

We can now apply Theorem 4.1 with $\epsilon = r\delta$ to show that there exists a choice of each v_i such that

$$(1 + \sqrt{r\delta})^2 \geq \left\| \sum_{i=1}^n v_i v_i^* \right\| = \left\| \sum_{j=1}^r \sum_{i \;:\; v_i = w_{i,j}} (\sqrt{r}w_{i,j})(\sqrt{r}w_{i,j})^* \right\|.$$

Letting $A_j := \{i : v_i = w_{i,j}\}$ for each $1 \leq j \leq r$, we get

$$\left\| \sum_{i \in A_j} u_i u_i^* \right\| = \left\| \sum_{i \in A_j} w_{i,j} w_{i,j}^* \right\| \leq \frac{1}{r} \left\| \sum_{j=1}^r \sum_{i \;:\; v_i = w_{i,j}} (\sqrt{r}w_{i,j})(\sqrt{r}w_{i,j})^* \right\| \leq (\frac{1}{\sqrt{r}} + \sqrt{\delta})^2$$

for all $1 \leq j \leq r$. $\square$

Then, choosing $r = 2$, $\delta = \frac{1}{18}$, implies KS_2 with $N = 18$ and $\epsilon = 2$. Thus the Kadison-Singer Problem, and its many equivalent formulations, are true.

5 Further Research and Applications

The proof of the Kadison-Singer Problem led to the resolution of many other conjectures, some of which we mention here.

Theorem 5.1 (Casazza-Tremain [16]). *Every unit norm 18-tight frame can be partitioned into two subsets, each of which has frame bounds 2,16.*

Theorem 5.2 (Sundberg Problem [15]). *Every bounded Bessel sequence can be written as the finite union of non-spanning sets.*

We also have applications in harmonic and Fourier analysis. For example,a result due to Lawton [27], for which we now introduce the necessary terminology.

Definition 5.3. A set $S \subset \mathbb{N}$ is called ***syndetic*** if, for some finite subset $F \subset \mathbb{N}$ we have

$$\cup_{n \in F}(S - n) = \mathbb{N}$$

where

$$S - n = \{m \in \mathbb{N} : m + n \in S\}.$$

Syndetic sets have ***bounded gaps***. There is an integer p, called the ***gap length***, such that $\{a, a+1, \ldots, a+p\} \cap S \neq \emptyset$ for every $a \in \mathbb{N}$.

Theorem 5.4. *The Fourier frame $\{e^{2\pi int}\chi_E\}_{n\in\mathbb{Z}}$ for $L^2(E)$ can be partitioned into r syndetic sets $A_1, \ldots, A_r$ with gap length $p \leq r$ so that $\{e^{2\pi int}\chi_E\}_{n\in A_j}$ is an ϵ-Riesz sequence for all $1 \leq j \leq r$.*

A few other new theorems arose from the proof of the Kadison-Singer problem. The following ideas regarding ***large*** Hilbert spaces were introduced in [10], leading to a result by Casazza and Tremain in [13].

Definition 5.5. A subspace $\mathcal{H} \subset \ell_2$ is can ***A-large*** if, for $A > 0$, it is closed and for each $i \in \mathbb{N}$, there is a vector $v_i \in \mathcal{H}$ so that $\|v_i\| = 1$ and $|v_i(i)| \geq A$, where $v(i)$ is the i^{th} component of v.

Definition 5.6. A closed subspace $\mathcal{H} \subset \ell_2$ is ***r-decomposable*** if for some $r \in \mathbb{N}$, there exists a partition $A_1, \ldots, A_r$ of $\mathbb{N}$ so that $P_{A_j}(\mathcal{H}) = \{(a_1, a_2, \ldots) \in \ell_2 : a_i = 0 \text{ if } i \notin A_j\}$ for all $1 \leq j \leq r$.

Theorem 5.7. *For every $0 < A < 1$ and $0 < \epsilon < 1$, there exists $r \in \mathbb{N}$ such that every A-large subspace of ℓ_2 is r-decomposable.*

Another consequence of Marcus, Spielman and Srivastava's proof of the Kadison-Singer Problem is that, in every theorem we have discussed, there is a construction to find the mentioned constants. For example, in Theorem 5.7, we have that

$$r = \left(\frac{6(A^2+1)}{\epsilon A^2}\right)^m$$

where $m = 4$ when working over $\mathbb{R}$ and $m = 8$ when working over $\mathbb{C}$.

5.1 Open Problems

A number of open problems remain, despite the work of Marcus, Spielman and Srivastava. We take these primarily from the questions raised in [13].

Problem 5.8. *Can N and ϵ in KS_r be improved?*

Problem 5.9. *Can the values of r in the various results be improved?*

Problem 5.10 ([16][17])**.** *Can every unit norm 2-tight frame be partitioned into three subsets, each of which are Riesz basic sequences, with Riesz bounds independent of the dimension of the ambient space?*

Problem 5.11. *Is there an implementable algorithm for proving the Paving Conjecture?*

Problem 5.12 ([10], [11])**.** *Does the Paving Conjecture for Toeplitz operators hold?*

Problem 5.13 ([10])**.** *Does there exist $\epsilon > 0$ so that, for large K, for all n and all equal norm 1-tight frames $\{f_i\}_{i=1}^{Kn}$ for ${\ell_2}^n$, there is a $J \subset \{1, 2, \ldots, Kn\}$ so that both $\{f_i\}_{i\in J}$ and $\{f_i\}_{i\in J^c}$ have lower frame bounds which are greater than ϵ?*

Clearly, there remains a significant amount of work in the wake of the Kadison-Singer Problem's solution.

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