Toric Varieties of Brick Polytopes, Associahedra, and Operad Structures

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1. Introduction

In a 2014 paper by Laura Escobar [2], she provides a lovely link between the theory of subword complexes, which were introduced by Knutson and Miller to study the combinatorics of Schubert polynomials, and varieties, which are objects of interest in algebraic geometry. She describes a construction of a subvariety of the Bott-Samelson variety associated to an element of a Coxeter group, which also defines a subword complex. She goes on to show that the dual polytope of the subword complex is the polytope associated to the variety constructed.

She then describes a particular choice of elements such that the polytope found is a realization of a polytope occurring frequently in combinatorics, the **associahedron**. The associahedron can be realized in many different ways; Escobar finds the realization due to Loday [4]. Our goal is to find and examine the properties of the varieties whose polytope is the realization of the associahedron due to Chapoton-Fomin-Zelevinsky [1].

3. Subword Complexes in Coxeter Groups

Definition. A **Coxeter group** is a group with generating set $S := \{s_i : i \in I\}$, identity id, and a map $m : S \times S \to \mathbb{N}$ such that:

1. $m(s_i, s_i) = 2 \ \forall i \in I$.

2. $(s_i s_j)^{m(s_i,s_j)} = \text{id} \ \forall i, j \in I$.

For our purposes, we shall only consider the symmetric group, which is clearly a Coxeter group. We now define a **subword complex**. A **word** in S is an ordered sequence $Q = (q_1, q_2, \ldots, q_n)$ of elements of S, $q_i \in S$. We call $J = (r_1, r_2, \ldots, r_n)$ a **subword** of Q if J can be obtained from Q by replacing some of the letters by the empty character $-$. We say a word Q contains $w \in W$ if the ordered product of generators in some subword of Q equals w. The subword complex of a word Q and and element w is the simplicial complex of subwords J of Q, such that the complement $Q \setminus J$ contains w. To every subword complex, we can associate a **brick polytope**, defined at the convex hull of a set of vectors determined by the faces of our complex. For example, if $Q = (s_1, s_2, s_1, s_2, s_1)$ and $w = s_1 s_2 s_1$, then the simplicial complex $\Delta(Q, w)$, and its brick polytope are:
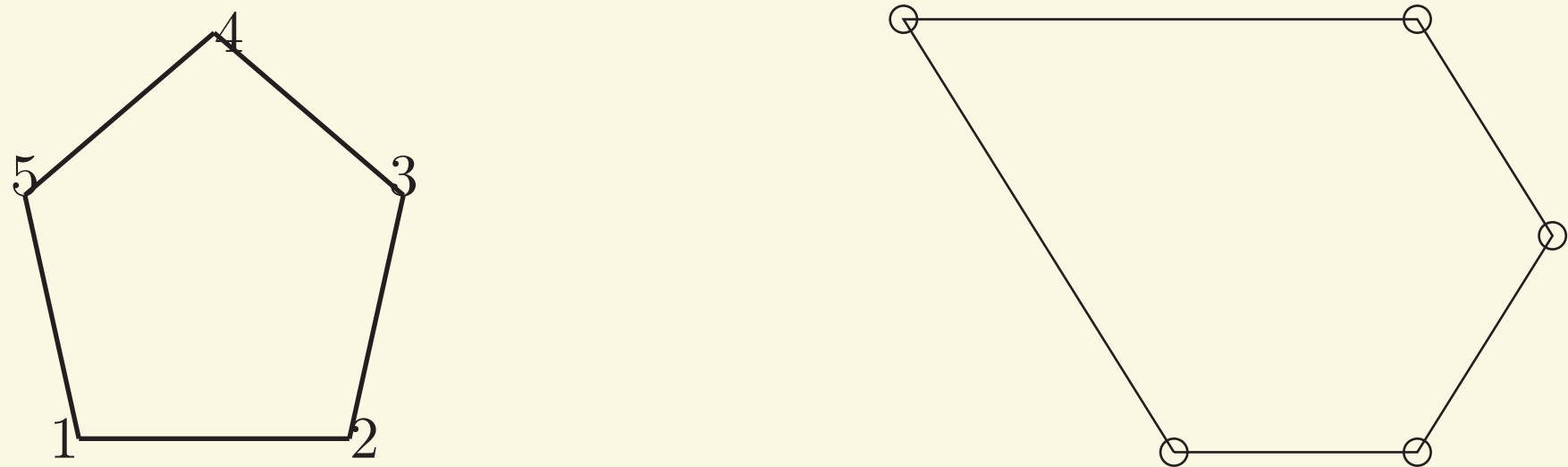
5. Realizing the Associahedron

Definition. The n-th **Tamari lattice** is a partially ordered set, denoted T_n, in which the elements consist of different ways of grouping a sequence of $n + 1$ objects into pairs using brackets, with $A \leq B$ if B may be obtained only by rightward applications of the associative law $(xy)z = x(yz)$.

The **associahedron** K_n is an $(n - 2)$-dimensional convex polytope whose 1-skeleton, its vertex and edge set, is the Tamari lattice T_{n-1}. As this definition is fairly loose, we in fact have infinitely many polytopes we can call K_n, including an infinite family of lattice polytopes due to Hohlweg-Lange, called realizations of the associahedron. Two realizations with simple descriptions are those due to Loday and Chapoton-Fomin-Zelevinsky. For example, these realizations of K_5 are
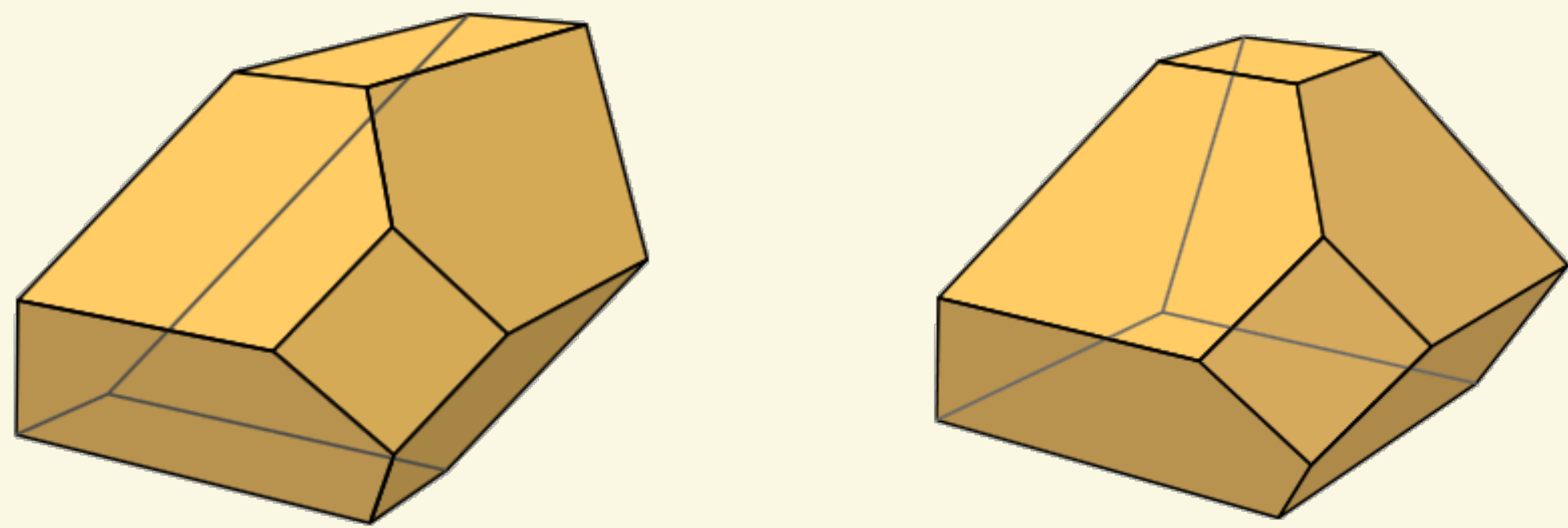
To be precise, the Loday associahedron K_{n+1} is defined as the convex hull of the vectors

$$v_T = (a_1 b_1, \ldots a_n b_n)$$

where T ranges over all binary trees with $n+1$ leaves and, letting i be the vertex between the $(i-1)$-th and i-th, where a_i is the number of leaves in the left subtree of i and b_i is the number of leaves in the right subtree.

The Chapoton-Fomin-Zelevinsky associahedron, however, is defined as dual to a cluster complex, but has a simple description in terms of it's normal fan. The facet normals of the CFZ associahedron are

$$\{e_i - e_j : 1 \leq i < j \leq n\} \cup \{e_{i+1} - e_i : 1 \leq i \leq n - 1\}$$

from which we can construct the polytope.

7. Operads and Associahedra

Perhaps one of the more intrigueing aspects of Escobar's "Loday" varieties is the existence of an operad structure, as found by Dotsenko, Shadrin and Vallette [?]. A **non-symmetric operad** is a collection $\{P(n)\}_{n=1}^{\infty}$ of "n-ary operations", along with a collection binary operations called **infinitesimal composition at slot** i:

$$\circ_i : P(m) \times P(n) \to P(m + n - 1) \ i = 1, \ldots, m$$

satisfying certain compatibility axioms.

One might hope that we could obtain a similar structure of the space of CFZ varieties. While we obtain many similar properties, such as a stratification by planar trees and an operad structure on the space of 0-dimensional strata, no analoguous structure seems to exist.

References

[1] F. Chapoton, S. Fomin, and A. Zelevinsky. Polytopal realizations of generalized associahedra. *Canadian Mathematical Bulletin*, 45:537–566, 2002.

[2] L. Escobar. Brick manifolds and toric varieties of brick polytopes. *Discrete Math. Theor. Comput. Sci. Proc*, pages 863–874, 2014.

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[4] J.-L. Loday. Realization of the Stasheff polytope. *Archiv Math*, 83:267–278, 2004.

[5] V. Pilaud and C Stump. Brick polytopes of spherical subword complexes and generalized associahedra. *To appear in Advances in Mathematics*, 2011.

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2. What is a Toric Variety

The simplest definition of a **variety** over $\mathbb{C}^n$ or $\mathbb{CP}^{n-1}$ is the set of common zeros of an ideal in $\mathbb{C}[x_1, \ldots, x_n]$. More abstract varieties can be constructed by "glueing" together simple varieties in an appropriate fashion. We call a variety **irreducible** if cannot be written as a union of its subvarieties. A class of particular interest are **toric** varieties. These are irreducible varieties satisfying:

1. $(\mathbb{C}^*)^n$ is a Zariski open subset of X.

2. The action of $(\mathbb{C}^*)^n$ on itself extends to an action on X.

where the Zariski topology given by $V \subset X$ is open if $X \setminus V$ is a subvariety of X. A method of constructing toric varieties uses convex polyhedral cones and fans of such cones. Quite a large class can be constructed in this way, called normal varietes, and it includes all those varieties of interest to us. For example, the variety $\mathbb{CP}^2$ arises from the fan
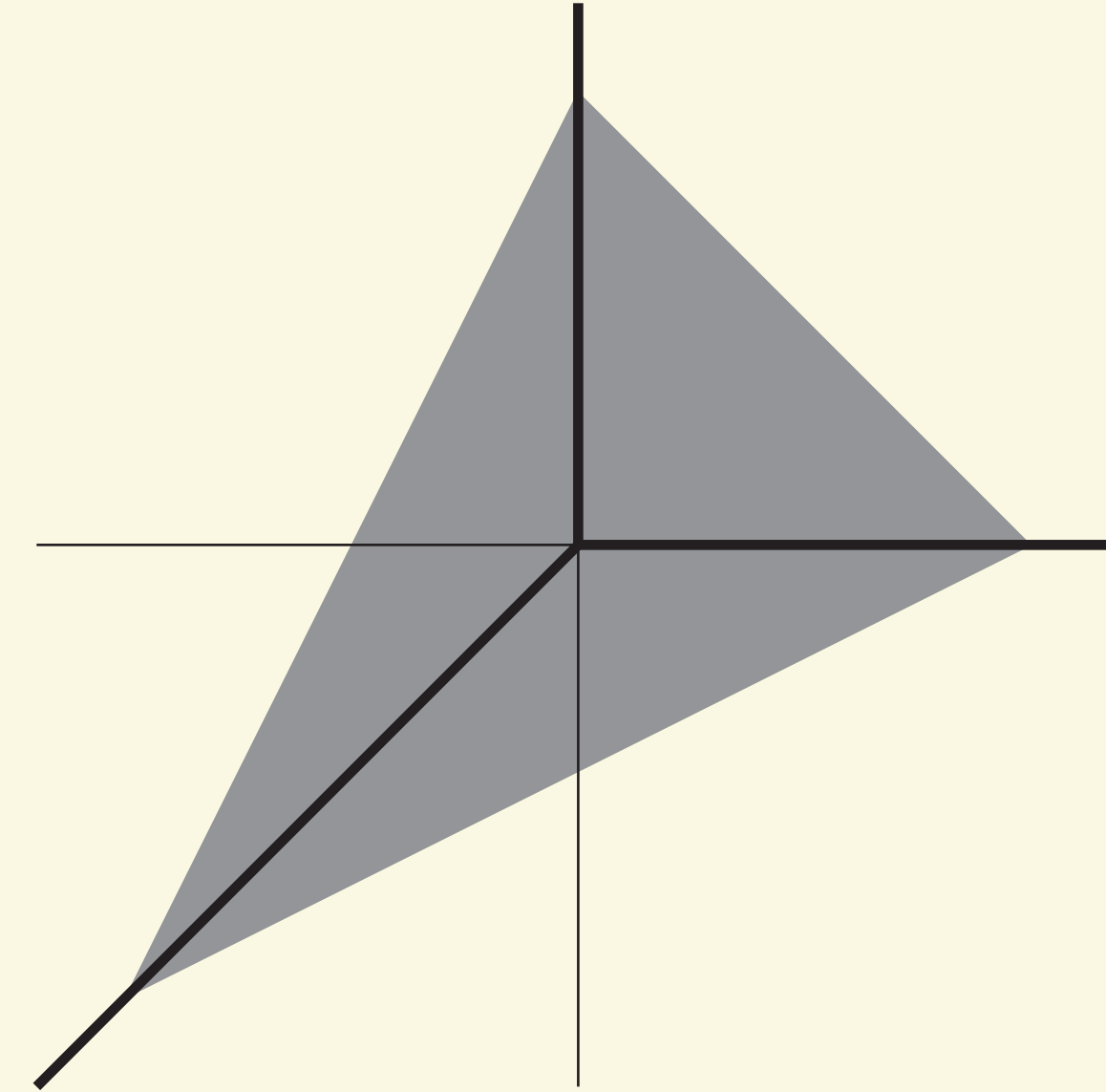
where heavy lines represent one dimensional cones, and shaded regions represent two dimensional cones. These fans can be viewed as normal fans of convex polytopes, allowing us to associate to every normal toric variety a polytope.

4. The Brick Manifold of Escobar

In [2], Escobar describes a very general construction of a toric variety. We shall briefly describe the case involving the symmetric group. We define the **base flag** to be

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_{n-1} \rangle \subset \mathbb{C}^n$$

where $e_1, \ldots, e_n$ is the standard basis og $\mathbb{C}^n$. Then, given a word $Q = (q_1, \ldots, q_m)$ in the generators $S = \{s_i = (i \ i+1) : 1 \leq i \leq n-1\} \subset S_n$, we define the Bott-Samelson variety BS^Q to be the space of all collections of flags flags $(F_1, \ldots F_m)$ such that F_1 is the base flag, and F_i and F_{i+1} differ only at one vector space, of dimension k where $q_i = s_k$. We define the **brick manifold** $B(Q, n+1)$ to be the subvariety of BS^Q such that F_m is the reverse flag

$$\langle e_n \rangle \subset \langle e_{n-1}, e_n \rangle \subset \cdots \subset \langle e_2, \ldots, e_n \rangle \subset \mathbb{C}^n$$

For example $B(s_1 s_2 s_1 s_2 s_1, 4)$ is the variety whose points consist of $\{V_1, V_2, V_3\}$ such that the following inclusion diagram holds:
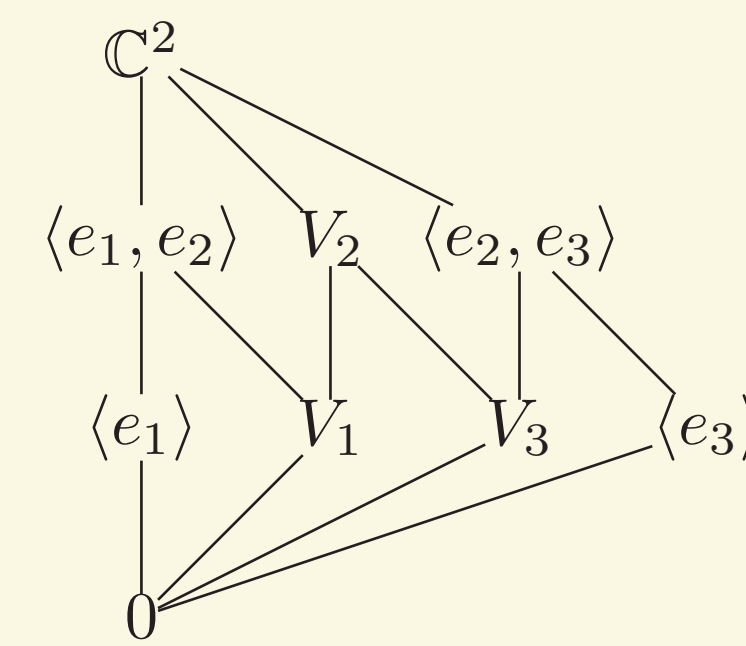
One result regarding this construction is the following theorem:

Theorem. $B(Q, n + 1)$ is the toric variety associated to the brick polytope of $\Delta(Q, w_0)$, where w_0 is the longest permutation in S_n.

As a consequence of this, the polytope of the above variety is the Loday associahedron and in general, for $Q = (s_1, s_2, \ldots s_{n-1}, s_1, \ldots, s_{n-1}, \ldots, s_1, s_2, s_1)$, we obtain the Loday associahedron.

6. Realizing Chapoton-Fomin-Zelevinsky

Building on the work of Escobar, we were able to determine a choice of Q such that the corresponding polytope is the Chapoton-Fomin-Zelevinsky associahedron, of type A_n. It is obtained by grouping odd-indexed and even-indexed generators in Q. For example, K_5 is realized by $Q = (s_1, s_3, s_2, s_1, s_3, s_2, s_1, s_3, s_2)$, with corresponding variety consisting of acceptable fillings of the diagram:
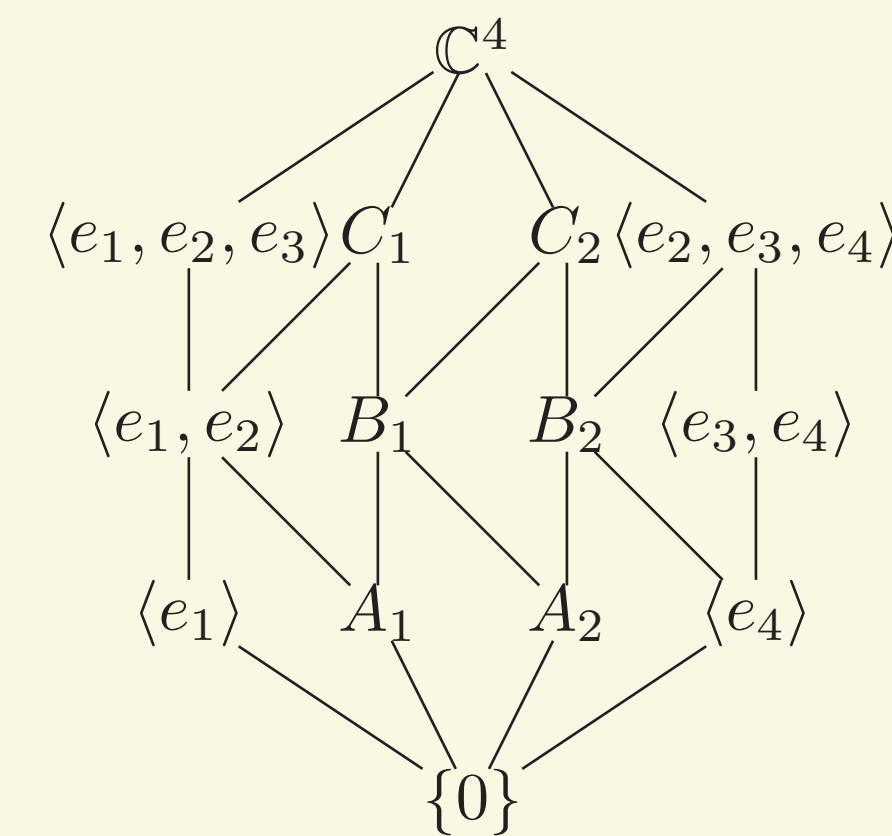
We call such varieties **CFZ varieties**. In this project, we prove that this result holds generally, using the theory of sorting networks as developed by Pilaud and Pocchiola.

8. A Few Conjectures

We would still like to achieve a few things, such as realizing generalized associahedra as subword complexes, and investigating the properties of their associated toric varieties. We know this to be possible, based on the work of [5], but the details remain unclear. We would also like to see if we can define an operad structure on a space containing the CFZ varieties. Based on correspondence with Chapoton and Pilaud, we do not believe that the space of CFZ varieties form an operad, but we are optimistic that an operad containing the CFZ varieties, along with the Loday varieties as a suboperad, exists.