Toric Varieties of Brick Polytopes, Associahedra and Operad Structures

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Abstract

In this project, we examine realizations of certain classes of toric varieties as brick manifolds, defined by Escobar. To do so, we use the language of subword complexes, introduced by Knutson and Miller. Pilaud and Stump realized certain subword complexes as dual to a convex polytope. In particular, for certain complexes, and their corresponding varieties, in the work of Escobar, we find that this convex polytope is a realization of the associahedron. Escobar provides a realization as the Loday associahedron, while we will provide a realization as the Chapoton-Fomin-Zelevinsky associahedron of type A_n. We also give a stratification of the corresponding variety in terms of binary tree. In addition, we describe the operad of varieties defined by Dotsenko, Shadrin and Vallette, before discussing why we believe an analogous operad does not exist. Finally, we conjecture possible realizations of the Chapoton-Fomin-Zelevinsky associahedron of type B_n, along with the existence of an operad containing that of Dotsenko, Shadrin and Vallette, along with the analogous structure we wished to define.

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1 Introduction

As in any good introduction to combinatorics, we start by defining the Catalan numbers, an extremely common count for the number of combinatorial objects in a certain class. The Catalan numbers are defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

and count a seemingly endless list of objects: triangulations, planar rooted binary trees, Dyck paths, lattice paths... When the Catalan numbers appear, we can sometimes endow the set of counted objects with the structure of a poset, called the Tamari Lattice [31]. For example, a way of defining the lattice is with vertices given by ways of bracketing expressions in n variables so that their multiplication is unambiguous and edges determined by associativity rules. This lattice forms the edge set of a convex polytope known as the associahedron [19].

As the associahedron is only defined by its vertex and edge sets, there are infinitely many ways of realizing it as a convex polytope [14], [13], [20], [4]. One description of the associahedron is through subword complexes. Subword complexes are algebraic structures, based on the simplicial complexes of algebraic topology, that tie together the combinatorial features of certain groups. They were introduced in [18] by Knutson and Miller to study the combinatorics of Schubert polynomials, but soon flourished in their own right.

Realizing the associahedron via subword complexes proved to have an additional interesting structure. In [10], Escobar defines a class of toric varieties associated to subword complexes and realizing the associahedron.This class then caught the eye of Dotsenko, Shadrin and Vallette,who endowed it with the structure of an operad [8]. However, in [8], they only examine the Loday associahedron. We shall discuss the toric varieties realizing the associahedron of Chapoton-Fomin-Zelevinsky.

Theorem. 5.18 *The Chapoton-Fomin-Zelevinsky associahedron of type A_n is dual to the subword complex $\Delta(Q, w_0)$, where w_0 is the longest element of S_n and Q is determined by the Coxeter element $s_1 s_3 \ldots s_2 s_4 \ldots$*

The structure of our discussion will be as follows. In Section 2, we develop much of the necessary background in algebraic geometry before developing the theory of subword complexes and the construction of Escobar in Section 3. We will then discuss the associahedron in more detail in Section 5, focusing particularly on the realization due to Chapoton-Fomin-Zelevinsky, before realizing it as a subword complex using the theory of sorting networks [24]. We also discuss briefly the structure of the corresponding variety. In Sectionoperad we discuss the construction of this bigger algebraic structure as in [8], before attempting to recreate it via a stratification of the Chapoton-Fomin-Zelevinsky varieties. Finally we leave the reader with a few conjectures and questions in Section 7.

2 Some definitions

2.1 Algebraic varieties

In this section, we aim to give the reader a brief description of the necessary algebraic geometry needed in later discussion. It is by no means a comprehensive guide and we invite the reader to look to [28] or any other introductory text for more information.

We start be defining an affine variety. While these can be defined over any field, we shall, for sake of convenience, work over $\mathbb{C}$.

Definition 2.1. Given polynomials $f_1, f_2, \ldots, f_s \in \mathbb{C}[x_1, x_2, \ldots, x_n]$, the **affine variety** associated to these polynomials is

$$V(f_1, \ldots, f_s) := \{a \in \mathbb{C}^n : f_1(a) = f_2(a) = \cdots = f_s(a) = 0\}.$$

For an ideal $I \subset \mathbb{C}[x_1, x_2, \ldots, x_n]$, we define $V(I) := \{a \in \mathbb{C}^n : f(a) = 0 \ \forall f \in I\}$.

Definition 2.2. Given a variety $V \subset \mathbb{C}^n$ we define it's corresponding ideal by

$$I(V) := \{f \in \mathbb{C}[x_1, x_2, \ldots, x_n] : f(a) = 0 \ \forall a \in V\}.$$

We also define the **coordinate ring** of V to be the space of polynomials on V. It is isomorphic to $\mathbb{C}[V] := \mathbb{C}[x_1, x_2, \ldots, x_n]/I(V)$.

Example 2.3. The variety $V = \{(x, y) \in \mathbb{C}^2 : y = x^2\}$ is the variety generated by the polynomial $f(x, y) = y - x^2$. We have $I(V) = (y - x^2)$ and $\mathbb{C}[V] = \mathbb{C}[x, y]/(y - x^2) \simeq \mathbb{C}[x]$.

Any finitely generated $\mathbb{C}$-algebra R with no nilpotents can be considered as the coordinate ring of an affine variety, $V = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_n]/I)$, where I is a radical ideal, by identifying V with maximal ideals of the quotient.

Lemma 2.4. *Given an affine variety V, we have $V = Spec(\mathbb{C}[V])$.*

We can also define a topology on a variety using the **Zariski** topology: the open subsets of a variety V are $V \setminus W$ where W is a subvariety of V. Given a set $S \subset V$, its closure $\bar{S}$ in the Zariski topology is the smallest subvariety of V containing S.

Lemma 2.5. *In $\mathbb{C}^n$, $\bar{S} = V(I(S))$.*

In order to simplify discussions later on, we will primarily consider irreducible and normal varieties.

Definition 2.6. An affine variety is called **irreducible** if it cannot be written as the union of subvarieties.

Lemma 2.7. *The following are equivalent:*

1. *V is an irreducible variety.*

2. *$I(V)$ is a prime ideal.*

3. $\mathbb{C}[V]$ *is an integral domain.*

Definition 2.8. An integral domain R is called **integrally closed** if every element of its field of fractions that is integral over R is in R.

We call an irreducible variety V **normal** if $\mathbb{C}[V]$ is integrally closed.

We can extend the definition of a variety to projective space as well, and all mentioned results and definitions are easily transferred across from the affine case. We denote by $\mathbb{P}^n$ n-dimensional projective space, parametrized by $n+1$ complex co-ordinates.

Definition 2.9. Given $f_1, f_2, \dots, f_s \in \mathbb{C}[x_1, \dots, x_n, x_{n-1}]$, homogeneous polynomials, we define their corresponding **projective variety** by

$$V(f_1, f_2, \dots, f_s) := \{a\mathbb{P}^n : f_1(a) = \cdots = f_s(a) = 0\}.$$

Unfortunately, projective varieties do not encompass all the objects we can investigate using the methods of algebraic geometry. We must still introduce a more abstract notion of variety, obtained by "glueing" together affine varieties.

Definition 2.10. An **abstract variety** X is defined by a collection $(\{V_\alpha\}_{\alpha\in I}, \{V_{\alpha,\beta}\}_{\alpha,\beta\in I}, \{g_{\alpha,\beta}\}_{\alpha,\beta\in I})$ where

1. V_α is an affine variety.
2. $V_{\alpha,\beta} \subset V_\alpha$ are Zariski open.
3. $g_{\alpha,\beta} : V_{\alpha,\beta} \simeq V_{\beta,\alpha}$ are isomorphisms sutch that $g_{\alpha,\alpha} = \mathrm{id}_{V_\alpha}$ and $g_{\beta,\gamma} \restriction_{V_{\beta,\alpha}\cap V_{\beta,\gamma}} \circ g_{\alpha,\beta} \restriction_{V_{\alpha,\beta}\cap V_{\alpha,\gamma}} = g_{\alpha,\gamma} \restriction_{V_{\alpha,\beta}\cap V_{\alpha,\gamma}}$.

The abstract variety is then given by $X = \cup_{\alpha\in I} V_\alpha / \sim$ where $a \in V_\alpha \sim b \in V_\beta$ if $a \in V_{\alpha,\beta}$ and $b = g_{\alpha,\beta}(a)$.

Example 2.11. Consider the affine variety with

$$V_1 = V_2 = V_3 = \mathbb{C} \times \mathbb{C} \setminus \{0\}$$

$$V_{i,j} = \mathbb{C} \setminus \{0\} \times \mathbb{C} \setminus \{0\}$$
$$g_{i,j}(x, y) = (x^{-1}, y^{-1})$$

This gives three copies of $\mathbb{C}^2$ without the origin, glued pairwise along $\mathbb{C} \setminus \{0\}$, which is precisely $\mathbb{P}^2$, the projective plane.

Finally we define the blowup of a variety at a point. Blowing up a variety at a point, or even along an entire subvariety provides a way of obtaining new varieties and also of resolving singularities. Again, we refer the reader to any standard text on algebraic geometry for further details. In our discussion, blowups will only arise as the result of later constructions and so we need only to be able to recognize them.

Definition 2.12. Let X be an n-dimensional variety. Consider the space $\{(q, \vec{q}) \in X \times \mathbb{P}^{n-1}\}$, where $\vec{q}$ is the line passing through q and the origin, and the map $\pi : \{(q, \vec{q}) \in X \times \mathbb{P}^{n-1}\} \to X$ given by $\pi(q, \vec{q}) = q$. The **blowup** of a variety $V \subset X$at a point p is defined to be the Zariski closure in $X \times \mathbb{P}^{n-1}$ of $\pi^{-1}(V \setminus p)$.

We denote the blowup of a variety at p by $Bl_p(V)$.

Example 2.13. The blowup of $\mathbb{C}^2$ at the origin is given by $V(x_1y_2 - y_1x_2)$ where x_1, x_2 are coordinates on $\mathbb{C}^2$ and y_1, y_2 are coordinates on $\mathbb{P}^1$. This gives a variety that is identical to $\mathbb{C}^2$ everywhere except the origin. At the origin of $\mathbb{C}^2$, we obtain a copy of the projective line.

2.2 Toric varieties and polyhedral cones

A class of interesting varieties are those known as toric varieties. They come equipped with a torus action, and often have a nice geometric presentation, which will be the study of much of this project. Once again, we are unable to provide a comprehensive discussion, but we recommend the reader turn to [5], or [6], for more details, as we follow the former quite closely. As it will appear frequently in the following section, we will denote $\mathbb{C} \setminus \{0\}$ by $\mathbb{C}^*$.

Definition 2.14. A **toric** variety is an irreducible variety X such that

1. $(\mathbb{C}^*)^n$ is a Zariski open subset of X.
2. The action of $(\mathbb{C}^*)^n$ on itself extends to an action on X.

We call $(\mathbb{C}^*)^n$ the **torus** of X. We will now give some simple, possibly trivial, examples of toric varieties.

Example 2.15. Clearly $\mathbb{C}^*$ and $\mathbb{C}$ are examples of toric varieties. We can also view $\mathbb{P}^n$ as a toric variety: suppose that $x_0, x_1, \ldots, x_n$ are homogeneous coordinates on $\mathbb{P}^n$. We can then embed $(\mathbb{C}^*)^n$ into $\mathbb{P}^n$ by the map

$$(t_1, t_2, \ldots, t_n) \to (1, t_1, t_2, \ldots, t_n)$$

which identifies $(\mathbb{C}^*)^n$ with the Zariski open subset $\mathbb{P}^n \setminus V(x_0x_1 \ldots x_n)$. Furthermore, we can extend the torus action by letting

$$(t_1, t_2, \ldots, t_n)\dot{(}a_0, a_1, a_2, \ldots, a_n) := (a_0, t_1a_1, t_2a_2, \ldots, t_na_n)$$

showing that $\mathbb{P}^n$ is a toric variety.

Example 2.16. A slightly more interesting example is the cubic with a cusp $V(y^2 - x^3)$. We can embed $\mathbb{C}^*$ as a Zariski open subset of this by the map $t \mapsto (t^2, t^3)$ with torus action $t\dot{(}x, y) := (t^2x, t^3y)$. Thus $V(y^2 - x^3)$ is a toric variety.

One method of constructing toric varieties is through the theory of fans and cones. This method produces normal varieties, and gives us an interesting geometric take on these varieties.

Definition 2.17. A **convex polyhedral cone** in $\mathbb{R}^n$ is a subset

$$\sigma = \text{Cone}(S) := \left\{ \sum_{v \in S} \lambda_v v : \lambda_v \geq 0 \right\}$$

where $S \subset \mathbb{R}^n$ is finite. The dimension of σ is the dimension of the smallest subspace $\mathbb{R}\sigma \subset \mathbb{R}$ containing σ. We call $\mathbb{R}\sigma$ the span of σ.

Note that such a cone is indeed convex: if $x, y \in \sigma$ then $tx + (1-t)y \in \sigma$ for $0 \leq t \leq 1$.

Definition 2.18. If σ is a convex polyhedral cone, we define it's **dual cone** by

$$\sigma^\vee := \{u \in (\mathbb{R}^n)^* : \langle u, v \rangle \geq 0 \; \forall v \in \sigma\}.$$

Denoting by $H_u^+ := \{v \in \mathbb{R}^n : \langle u, v \rangle \geq 0\}$ the upper half space with respect to u, we note that $\sigma \in H_u^+$ if and only if $u \in \sigma^\vee$. From this, we define faces of σ as follows.

Definition 2.19. A **face** of σ is a subset of $\mathbb{R}^n$ given by $H_u \cap \sigma$ where $u \in \sigma^\vee$ is non-zero and $H_u := \{v \in \mathbb{R}^n : \langle u, v \rangle = 0\}$. A **facet** of σ is a face of codimension 1.

The following results follow immediately.

Lemma 2.20. *Let $\sigma = Cone(S)$. Then:*

1. *Every face of σ is a convex polyhedral cone.*
2. *The intersection of two faces of σ is a face of σ.*
3. *A face of a face of σ is a face of σ.*

We can also show that, if $\tau_1, \tau_2, \ldots, \tau_s$ are facets of σ, with $\tau_i = H_{u_i} \cap \sigma$, then $\sigma = \cap_{i=1}^n H_{u_i}^+$ and $\sigma^\vee = \text{Cone}(u_1, \ldots, u_s)$. Knowing this, it quickly follows that $(\sigma^\vee)^\vee = \sigma$.

Definition 2.21. A convex polyhedral cone $\sigma \subset \mathbb{R}^n$ is called **strongly convex** if $\sigma \cap (-\sigma) = \{0\}$.

The next result, again, follows easily from definitions.

Lemma 2.22. *The following are equivalent:*

1. *$\sigma \subset \mathbb{R}^n$ is strongly convex.*
2. *σ contains no positive dimensional subspace.*
3. *$\{0\}$ is a face of σ.*
4. *$dim(\sigma^\vee) = n$.*

Finally, we define lattices and rational polyhedral cones, before providing the relationship between toric varieties and cones.

Definition 2.23. A **lattice** N is a free Abelian group of finite rank $N \simeq \mathbb{Z}^n$. The dual lattice is given by $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$.

Definition 2.24. For a lattice N with dual M,define

$$N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \simeq \mathbb{R}^n$$
$$M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \simeq (\mathbb{R}^n)^*.$$

$\sigma \subset N_{\mathbb{R}}$ is a **rational polyhedral cone** if $\sigma = \mathrm{Cone}(S)$ for a finite $S \subset N$.

Given a rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$, define

$$S_\sigma := \sigma^\vee \cap M.$$

One can easily see that S_σ is an additive Abelian semigroup, with $0 \in S_\sigma$ as the identity. Gordon's Lemma tells us that it is finitely generated.

We then define the semigroup algebra $\mathbb{C}[S_\sigma]$ as the $\mathbb{C}$-vector space with basis $\{\chi^m : m \in S_\sigma\}$ and multiplication defined by

$$\chi^m \chi^{m\prime} := \chi^{m+m\prime}$$

and the distributive law. We also have a multiplicative unit $\chi^0 = 1$ of $\mathbb{C}[S_\sigma]$.

As S_σ is finitely generated, we can choose generators $m_1, m_2, \ldots, m_r$, and take their corresponding algebra elements $\chi_i := \chi^{m_i}$, letting us write any element of $\mathbb{C}[S_\sigma]$ as a linear linear combination of terms $\chi_1^{n_1}\chi_2^{n_2}\cdots\chi_r^{n_r}$. Finally, choosing a $\mathbb{Z}$-basis of N, and identifying χ_i with t_i, we obtain an obvious inclusion

$$\mathbb{C}[S_\sigma] \subset \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$$

of $\mathbb{C}$-algebras. It follows that $\mathbb{C}[S_\sigma]$ is an integral domain. Thus, we can make the following definition.

Definition 2.25. Let $V_\sigma := \mathrm{Spec}(\mathbb{C}[S_\sigma])$ be the affine variety of $\mathbb{C}[S_\sigma]$ as an integral domain.

Theorem 2.26. *If $\sigma \subset N_{\mathbb{R}}$ is a strongly convex rational polyhedral cone, V_σ is a normal affine toric variety.*

Proof. See [5, Theorem 1.13]. □

This gives us a method of constructing normal affine toric varieties. However, there is a much stronger connection between such varieties and cones, as shown in the following theorem.

Theorem 2.27. *Let V be an affine toric variety. Then V is isomorphic to V_σ for some strongly convex rational polyhedral cone σ if and only if V is normal.*

Example 2.28. Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$ and let $\sigma = \text{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3)$. The facet normals of σ are

$$m_1 = (1, 0, 0), m_2 = (0, 1, 0), m_3 = (0, 0, 1), m_4 = (1, 1, -1)$$

and thus generate $\sigma^\vee$. In fact, taking $N = \mathbb{Z}^3$, they generate $S_\sigma = \sigma^\vee \cap \mathbb{Z}^3$. Then, letting

$$x = \chi^{m_1}, y = \chi^{m_2}, z = \chi^{m_3}, w = \chi^{m_4}$$

and noting that $m_1 + m_2 = m_3 + m_4$, one can easily see that

$$\mathbb{C}[S_\sigma] \simeq \mathbb{C}[x, y, z, w]/\langle xy - zw \rangle.$$

Therefore $V_\sigma \simeq V(xy - zw) \subset \mathbb{C}^4$.

2.3 Fans and abstract toric varieties

The above construction only gives us affine toric varieties. It is however, quite easy to generalize this to abstract toric varieties by introducing the idea of a fan.

Definition 2.29. Given a lattice N, a **fan** is a finite collection Σ of cones in $N_\mathbb{R}$ satisfying

1. Each $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone,
2. If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$.
3. If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of each.

A fan Σ encodes the necessary glueing information to assemble an abstract variety X_Σ from the affine varieties $V_\sigma, \sigma \in \Sigma$ as follows.

Given a fan Σ, let τ be a face of $\sigma \in \Sigma$. Then $\tau \subset \sigma$ induces an inclusion $\mathbb{C}[S_\sigma] \subset \mathbb{C}[S_\tau]$ which in turn induces a map $V_\tau \to V_\sigma$. Thus, given cones $\sigma, \sigma' \in \Sigma$ with common face $\sigma \cap \sigma'$, we get open immersions

$$V_{\sigma\cap\sigma'} \to V_\sigma$$
$$V_{\sigma\cap\sigma'} \to V_{\sigma'}.$$

Denoting the images of these maps by $V_{\sigma\sigma'}$ and $V_{\sigma'\sigma}$ respectively, we obtain an isomorphism

$$g_{\sigma\sigma'} : V_{\sigma\sigma'} \simeq V_{\sigma'\sigma}.$$

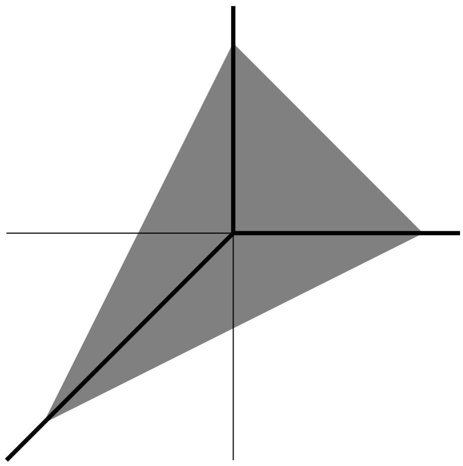
This give the necessary glueing data $\{V_\sigma, V_{\sigma\sigma'}, g_{\sigma\sigma'}\}$ as described in Definition 2.10.

Definition 2.30. Given a fan Σ in $N_\mathbb{R}$, X_Σ is the abstract variety constructed with the above glueing data.

Theorem 2.31. *X_Σ is a normal toric variety.*

In fact, in [16], it is shown that *every* normal toric variety arises in such a fashion. As such, given a normal toric variety, one can always find a fan Σ which gives rise to it. For more details regarding this, see [5].

Example 2.32. The following is the fan of $\mathbb{P}^2$, where the one dimensional cones are represented by heavy lines and the two dimensional cones are shaded.



To see that this fan gives rise to $\mathbb{P}^2$, note that each two dimensional cone is generated by a $\mathbb{Z}$-basis and thus corresponds to $\mathbb{C}^2$. Seeing how they fit together along the one dimensional cones gives rise to the usual construction of $\mathbb{P}^2$ from three copies of $\mathbb{C}^2$.

Dual to every fan is a polytope, allowing us to, in a sense, realize toric varieties as convex polyhedra. It is, however, easier to describe this duality starting with a polytope, so we shall start by defining that.

Definition 2.33. A **convex polytope** P is the set

$$P := \left\{ \sum_{v \in S} \lambda_v v : \lambda_v \geq 0 \text{ and } \sum_{v \in S} \lambda_v = 1 \right\}$$

where $S \subset \mathbb{R}^n$ is finite. We call P the **convex hull** of S.

The dimension of a polytope is defined, similarly to that of a cone, as the dimension of the smallest affine space containing it. A **face** of a polytope P is defined as $H_u \cap P$ where $H_u^+ \supset P$. An **facet** is a face of codimension 1 and a **vertex** is a face of dimension 0. We are concerned primarily with lattice polytopes.

Definition 2.34. Let $N \simeq \mathbb{Z}^n$ be a lattice with dual M. A **lattice polytope** $P \subset M_{\mathbb{R}} \simeq \mathbb{R}^n$ is the convex hull of a finite subset of M. We will assume, for the remainder of this section, that P has dimension n.

By assuming that $\dim P = n$, we have that the normal vector to a facet $F \subset P$ is unique up to multiplication by a scalar. Since the facet is defined over M, we can define a **facet normal** $\vec{n}_F \in N$ uniquely by requiring that $\vec{n}_F$ be primitive, so $\vec{n}_F$ is not an integer multiple of another element of N, and points into the interior of P.

Definition 2.35. Given a lattice polytope P and a face $F \subset P$, let σ_F be the cone in $N_{\mathbb{R}}$ generated by the facet normals of all facets containing F. The collection $\Sigma_P := \{\sigma_F : F \text{ is a face of } P\}$ forms a fan, called the **normal fan** of P. From this, we obtain a toric variety X_P associated to the polytope.

Clearly, this construction works in the other direction. Given a fan Σ, we can obtain a polytope

$$P_\Sigma := \cap_{u \in \Sigma} H^+_{u,a_u}$$

where we intersect over all one dimensional cones $u \in \Sigma$, $a_u \in \mathbb{Z}$ and

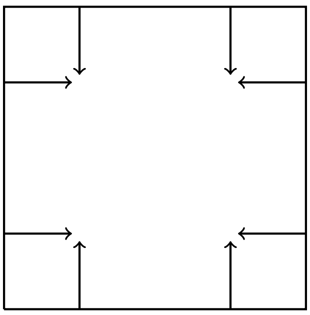
$$H^+_{u,a_u} := \{m \in M_{\mathbb{R}} : \langle m, u \rangle \geq -a_u\}.$$

P_Σ is dependent on choice of a_u, but that amounts to rescaling our basis and is therefore something we shall not dwell upon.

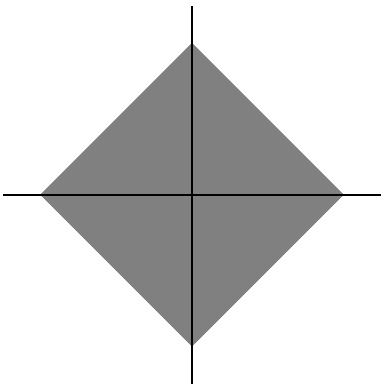
Example 2.36. The square with vertices $(0,0), (1,0), (0,1), (1,1)$ can be represented as

$$P = \{x \geq 0\} \cap \{y \geq 0\} \cap \{-x \geq -1\} \cap \{-y \geq -1\}.$$

This gives facet normal vectors $\pm e_1, \pm e_2$.



This gives rise to the fan



which we can show rise rise to the toric variety $X_P = \mathbb{P}^1 \times \mathbb{P}^1$.

3 Subword complexes and brick polytopes

3.1 Subword complexes in Coxeter groups

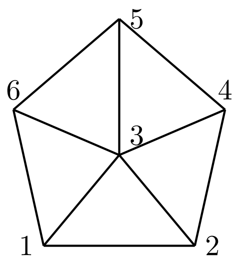
Definition 3.1. A **Coxeter group** is a group with generating set $S := \{s_i : i \in I\}$, identity id, and a map $m : S \times S \to \mathbb{N}$ such that:

1. $m(s_i, s_i) = 2 \ \forall i \in I$.
2. $(s_i s_j)^{m(s_i, s_j)} = \mathrm{id} \ \forall i, j \in I$.

Next we define a **word** in S to be an ordered sequence $Q = (q_1, q_2, \ldots, q_n)$ of elements of S, $q_i \in S$. We call $J = (r_1, r_2, \ldots, r_n)$ a **subword** of Q if J can be obtained from Q by replacing some of the letters by the empty character $-$. We define the complement of J in Q as having k-th entry $-$ if $r_k \neq -$ and k-th entry q_k otherwise. We denote the complement by $Q \setminus J$. Finally, given an element $w \in W$, we say that a word Q **contains** w if the ordered product of generators in some subword J of Q is equal to w. We can now define a subword complex, as introduced by Knutson and Miller in [18].

Definition 3.2. The **subword complex** of a word Q and an element w, is the simplicial complex of subwords F such that $Q \setminus F$ contains w. It has vertex set Q with facets F where $Q \setminus F$ is an expression of minimal length for w and is denoted by $\Delta(Q, w)$.

Example 3.3. If $Q = (s_1, s_2, s_3, s_1, s_2, s_1)$ and $w = s_1 s_2 s_1$, then the simplicial complex $\Delta(Q, w)$ is



Definition 3.4. We define the **Demazure product** of $w \in W$ and $s \in S$ b y

$$w \circ s = \begin{cases} ws & \text{if } \ell(ws) > \ell(w) \\ w & \text{if } \ell(ws) < \ell(w) \end{cases}$$

where $\ell(w)$ denotes the number of generators in a minimal expression for w, the **length** of w.
We define the Demazure product of a word $Q = (q_1, q_2 \ldots, q_n)$ to be

$$\mathrm{Dem}(Q) = (\cdots((s_1 \circ s_2) \circ s_3) \cdots \circ) \circ s_n.$$

In [17], Knutson and Miller further discuss the topology of subword complexes, showing that $\Delta(Q, w)$ is topologically a sphere if and only if $\mathrm{Dem}(Q) = w$. Throughout this paper, we shall assume that we are dealing with such choices of Q and w unless otherwise specified.

3.2 Brick polytopes

In [26], Pilaud and Stump define brick polytopes in terms of convex hulls of conjugates of fundamental weights of the Weyl group. We shall, however, use the definition of Escobar in [10]. The two definitions are known to be equivalent, as shown in Theorem 3.4 of [10]. Before we provide this definition, we must introduce some additional terminology.

Given a Coxeter group W, let $\Delta(W) := \{\alpha_s : s \in S\}$ be the simple roots of W and let $\nabla(W) := \{\omega_s : s \in S\}$ be the fundamental weights of W. Finally,for $J = (r_1, \dots, r_n)$ denote by $J_{(k)}$ the ordered product of the first k elements of J, considering $-$ as the identity element and defining $J_{(0)} = \mathrm{id}$.

Definition 3.5. Given a subword complex $\Delta(Q, w)$ we define the **root function**

$$r(J, \cdot) : \{\text{Subwords of } Q\} \to \Delta(W)$$

by

$$r(J, k) := (Q \setminus J)_{(k-1)}(\alpha_{q_k}).$$

Also define the **weight function**

$$w(J, \cdot) : \{\text{Subwords of } Q\} \to \nabla(W)$$

by

$$w(J, k) := (Q \setminus J)_{(k-1)}(\omega_{q_k}).$$

Definition 3.6. The **brick vector** of a face J of $\Delta(Q, w)$ is given by

$$\mathcal{B}(J) := \sum_{1 \leq k \leq |Q|} w(J, k).$$

and the **brick polytope** of $\Delta(Q, w)$ is defined as a convex hull:

$$\mathcal{B}(Q, w) := \mathrm{conv}\{\mathcal{B}(J) : J \in \Delta(Q, w) \text{ and } (Q \setminus J)_{(|Q|)} = w\}.$$

Definition 3.7. A word Q is **root independent** if, for some vertex $\mathcal{B}(J)$ of $\mathcal{B}(Q, w)$, we have that the multiset $r(J) := \{\{r(J, i) : i \in J\}\}$ is linearly independent.

In Pilaud and Stump's work [26], they show that if Q is root independent, the brick polytope is dual to the subword complex. Escobar builds on this result in [10], defining a brick manifold, which she showed to be a toric variety when Q is root independent, with the brick polytope as the associated polytope. We shall briefly introduce her construction and describe some of the motivating results for this work.

4 Brick manifolds and associated polyhedra

4.1 Brick manifolds over semisimple Lie groups

In this section, we introduce some results due to Escobar in their most general form, before focusing specifically on the case of the symmetric group.

Definition 4.1. A **Lie group** is a group that is also a differentiable manifold, in which the group operations of multiplication and inversion are smooth.

In our discussion, we shall assume that our Lie group G is complex and semisimple. We shall not define this, but rather refer the reader to any standard text on Lie groups and Lie algebras, [15] for example. As a result of these assumptions, we can consider G as a variety with the Zariski topology.

Definition 4.2. A **Borel subgroup** $B \subset G$ is a maximal Zariski closed and connected solvable subgroup.

A **torus** $T \subset G$ is a compact, connected, Abelian subgroup of G. The **Weyl group** of G with respect to T is the quotient $N(T)/Z(T)$ of the normalizer of T by the centralizer of T. The Weyl group is know to be a Coxeter group.

Given G, B and a maximal torus $T \subset B$, be let W be the Weyl group of G with respect to T, as defined above, with generators $S = \{s_1, s_2, \ldots, s_n\}$ and corresponding simple roots $\Delta(W) = \{\alpha_1, \ldots, \alpha_n\}$. A **parabolic** subgroup of G is a subgroup $P \supset B$. Denote by P_i the minimal parabolic subgroup corresponding to s_i. We then have that $P_i/B \simeq \mathbb{P}^1$ and that T has an action on this quotient with two fixed points.

Definition 4.3. Let $Q = (s_{i_1}, s_{i_2}, \ldots, s_{i_m})$ be a word in the generators of W. The product $P_{i_1} \times P_{i_2} \times \cdots P_{i_m}$ has an action of B^m given by

$$(b_1, b_2, \ldots, b_m)\dot{(}p_1, p_2, \ldots, p_m) := (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{m-1}^{-1} p_m b_m).$$

The **Bott-Samelson variety** of Q is the quotient

$$BS^Q := (P_{i_1} \times \cdots P_{i_m})/B^m.$$

Definition 4.4. We have a B equivariant map, $m_Q : BS^Q \to G/B$ defined by

$$m_Q(p_1, p_2, \ldots, p_m) := (p_1, \ldots, p_m)B.$$

We can now define the main object of interest for this section.

Definition 4.5. Let $Q = (q_1, q_2, \ldots, q_m)$ be a word in the generators of W and let $w = \mathrm{Dem}(Q)$. We define the **brick manifold** as the subset of BS^Q given by $m_Q^{-1}(wB)$.

Theorem 4.6. *Brick manifolds are smooth, irreducible varieties with dimension*

$$|Q| - \ell(w)$$

where $|Q|$ is the number of letters in Q.

Escobar proves the following very strong result, tying together brick polytopes and brick manifolds in [10].

Theorem 4.7. *Let $w = Dem(Q)$. The fibre $m_Q^{-1}(wB)$ is a toric variety with respect to T if and only if Q is root independent and $\ell(w) < |Q| \leq \ell(w) + \dim T$. Moreover, $m_Q^{-1}(wB)$ is the toric variety associated to the polytope $B(Q, w)$.*

As this has been quite abstract thus far, we will now discuss the case of $G = \mathrm{SL}_n(\mathbb{C}))$, for which the Bott-Samelson variety has a particularly nice realization.

4.2 Brick manifolds over $\mathbf{SL}_n(\mathbb{C})$

Definition 4.8. A **flag** in $\mathbb{C}^n$ is a series of vector subspaces

$$V_0 \subset V_1 \subset \cdots \subset V_r$$

where

$$0 \leq \dim V_0 < \dim V_1 < \cdots < \dim V_r \leq n$$

We call a flag **full** if $r = n$.

Let $G = \mathrm{SL}_n(\mathbb{C})$ and fix an ordered basis for $\mathbb{C}^n$. For our Borel subgroup, we choose B to the subgroup of $\mathrm{SL}_n(\mathbb{C})$ consisting of matrices that are upper triangular with respect to this basis. We then obtain an B-invariant flag,called the **base flag**:

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_{n-1} \rangle \subset \mathbb{C}^n$$

The maximal torus T in B is the subgroup consisting of all diagonal matrices in G. Let P_i be the minimal parabolic subgroup consisting of all matrices that are upper triangular, except possibly at $(P_i)_{i+1,i}$.

Lemma 4.9. *The quotient G/B can be identified with the flag variety*

$$\{\{0\} \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n : \dim V_i = i\}.$$

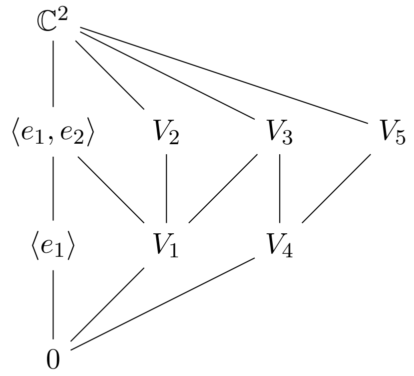
The Weyl group of G is $W = S_n$, with generators $S := \{s_1, \ldots, s_{n-1} : s_i = (i\ i+1)\}$. We can now define BS^Q for a word Q in S.

Definition 4.10. For $G = \mathrm{SL}_n(\mathbb{C})$ and $Q = (s_{i_1}, s_{i_2}, \ldots, s_{i_r})$, the Bott-Samelson variety BS^Q is realized as a sequence $(F_0, F_1, \ldots, F_{|Q|})$ of $|Q| + 1$ full flags, with F_0 the base flag, and F_k possibly differing from F_{k-1} only at the i_k dimensional subspace.

This is best illustrated with an example.

Example 4.11. Let $n = 3$ and $Q = (s_1, s_2, s_2, s_1, s_2)$. Then the Bott-Samelson variety is given by

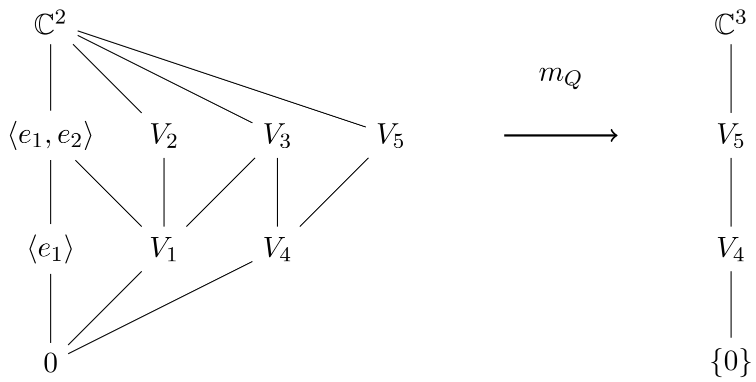
$$BS^Q = \{(V_1, V_2, V_3, V_4, V_5) : \text{The diagram below holds}\}$$



In general, we specify a point in BS^Q by giving subspaces $(V_1, \ldots, V_{|Q|})$ such that the incidence relations given by the flags hold. We can easily define a B- action on BS^Q, by having B act on the basis. We can then define a map $m_Q : BS^Q \to G/B$ that commutes with this action by

$$(F_0, F_1, \ldots, F_{|Q|}) \mapsto F_{|Q|}.$$

Example 4.12. Using the above example of $Q = (s_1, s_2, s_2, s_1, s_2)$, we have



This m_Q is indeed the desired map discussed in the previous section, so, letting $w = \mathrm{Dem}(Q)$, the brick manifold is given by $m_Q^{-1}(wB)$.

Escobar goes on to prove the following corollary, based of the work of Pilaud and Stump in [26].

Definition 4.13. Define a **Coxeter element** to be the product of all simple reflections, in some order, with each reflection appearing exactly once, represented by $\mathbf{c}$.

Given an element w in the Coxeter group, define the **c-sorting word** of w to be the lexicographically first subword of $\mathbf{c}^\infty$ that is a reduced expression for w, denoted $w(c)$.

We denote by w_0 the element with longest reduced expression. For S_n, $w_0(i) = n + 1 - i$.

Corollary 4.14. *Let $Q = (c)w_0(c)$ is the concatenation of a word $\mathbf{c}$ representing a Coxeter element c, and the $\mathbf{c}$-sorting word for w_0. Then $m_Q^{-1}(w_0 B)$ is the toric variety of the associahedron, as realized in [14] and [26].*

In particular, she notes in [9], that $\mathbf{c} = s_1 s_2 \cdots s_{n-1}$ gives rise to the Loday associahedron.

Example 4.15. Choosing $\mathbf{c} = s_1 s_2$, and $w_0 = (1\,3)$, we get $Q = \mathbf{c}w_0(c) = (s_1, s_2, s_1, s_2, s_1)$. The Bott Samelson variety is given by

$$BS^Q = \{(V_1, V_2, V_3, V_4, V_5) : \text{ The diagram below holds}\}$$

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & & \\
\langle e_1, e_2\rangle \arrow[d, dash] \arrow[dr, dash] & V_2 \arrow[d, dash] \arrow[dr, dash] & V_4 \arrow[d, dash] \arrow[dr, dash] & \\
\langle e_1\rangle \arrow[d, dash] & V_1 \arrow[dl, dash] & V_3 \arrow[dll, dash] & V_5 \arrow[dlll, dash] \\
0 & & &
\end{tikzcd}$$

Looking at the fibre corresponding to $m_Q^{-1}(w_0 B)$, we get that our variety, that we shall denote by $B(4) := \{(V_1, V_2, V_3) : \text{ The below diagram holds}\}$.

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & & \\
\langle e_1, e_2\rangle \arrow[d, dash] \arrow[dr, dash] & V_2 \arrow[d, dash] \arrow[dr, dash] & \langle e_2, e_3\rangle \arrow[d, dash] \arrow[dr, dash] & \\
\langle e_1\rangle \arrow[d, dash] & V_1 \arrow[dl, dash] & V_3 \arrow[dll, dash] & \langle e_3\rangle \arrow[dlll, dash] \\
0 & & &
\end{tikzcd}$$

We can describe this variety quite easily, by noting that choice of V_1 and V_3 uniquely determined V_2, unless $V_1 = V_3 = \langle e_2\rangle$. In that case, the variety is determined by a choice of a one dimensional subspace of $\langle e_1, e_3\rangle$. This precisely describes a point in the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at the origin. Thus $B(4) \simeq \mathrm{Bl}_0(\mathbb{P}^1 \times \mathbb{P}^1)$. As the polytope of $\mathbb{P}^1 \times \mathbb{P}^1$ is a square, from the theory of blowups and normal fans, as in [6, Proposition 3.3.15], we conclude the polytope of $\mathrm{Bl}_0(\mathbb{P}^1 \times \mathbb{P}^1)$ must be a pentagon.

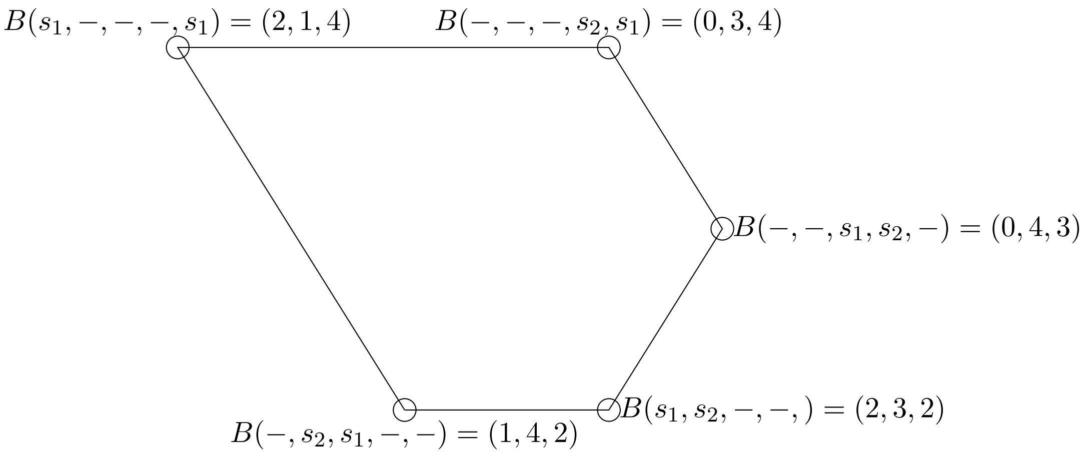
According to our corollary, this polytope should also be given by $B(Q, w_0)$, which we compute to be the convex hull of the vectors:

$$\begin{aligned}
B(s_1,-,-,-,s_1) &= \omega_1+\omega_2+s_2\omega_1+s_2s_1\omega_2+s_2s_1s_2\omega_1\\
B(s_1,s_2,-,-,-) &= \omega_1+\omega_2+\omega_1+s_1\omega_2+s_1s_2\omega_1\\
B(-,s_2,s_1,-,-) &= \omega_1+s_1\omega_2+s_1\omega_1+s_1\omega_2+s_1s_2\omega_1\\
B(-,-,s_1,s_2,-) &= \omega_1+s_1\omega_2+s_1s_2\omega_1+s_1s_2\omega_2+s_1s_2\omega_1\\
B(-,-,-,s_2,s_1) &= \omega_1+s_1\omega_2+s_1s_2\omega_1+s_1s_2s_1\omega_2+s_1s_2s_1\omega_1
\end{aligned}$$

where

$$\begin{aligned}
\omega_1 &= 2e_1-e_2-e_3\\
\omega_2 &= e_1+e_2-2e_3.
\end{aligned}$$

This leads to the polytope



which we will later see to be the Loday associahedron.

5 The associahedron and sorting networks

5.1 Realizations of the associahedron

We will now properly define the associahedron, something we have avoided thus far. We first introduce the Tamari Lattice, first introduced by Tamari [31].

Definition 5.1. The n-th **Tamari lattice** is a partially ordered set, denoted T_n, in which the elements consist of different ways of grouping a sequence of $n+1$ objects into pairs using brackets, with $A \leq B$ if B may be obtained only by rightward applications of the associative law $(xy)z = x(yz)$.

For example

$$(ab)c)d \leq (ab)(cd) \leq a(b(cd))$$

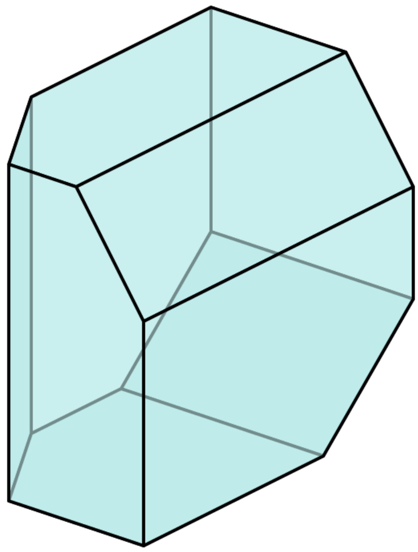
is a segment of the Tamari lattice. There are many more ways to describe the objects Tamari lattice: binary trees, triangulations of a polygon and almost any object counted by the Catalan numbers. See [29] for a non-exhaustive list.

Definition 5.2. An **associahedron** K_n is an $(n-2)$ dimensional convex polytope whose 1-skeleton, its vertex and edge set, is the Tamari lattice T_{n-1}.

Example 5.3. K_4 is any convex pentagon

$$\begin{tikzcd}[column sep=small]
 & a((bc)d) \arrow[dl, dash] \arrow[dr, dash] & \\
(a(bc))d \arrow[d, dash] & & a(b(cd)) \arrow[d, dash] \\
((ab)c)d \arrow[rr, dash] & & (ab)(cd)
\end{tikzcd}$$

K_5 is any convex polyhedron with the same 1-skeleton as the following polytope:



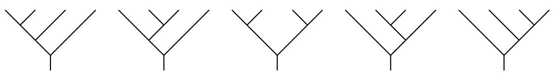
As there are many polytopes that can be considered associahedra, there have been a few constructions of particular interest, particularly those giving integer coordinates for each vertex. Two constructions of note were those of Loday and Chapoton-Fomin-Zelevinsky. We shall first describe the construction of Loday.

Theorem 5.4 ([20])**.** *Let Y_n be the set of binary trees with $n+1$ leaves. We label the leaves of a tree t $0, 1, \ldots, n$ from left to right and then label the internal vertices $1, 2, \ldots, n$ where the i^{th} vertex is the one which falls between the leaves $i-1$ and i. We denote by a_i the number of leaves to the left of vertex i, and by b_i the number of leaves to the right. Then to each tree $t \in Y_n$, we associate the point $M(t) \in \mathbb{R}^n$, where*

$$M(t) := (a_1 b_1, a_2 b_2, \ldots, a_n b_n).$$

The convex hull of $\{M(t) : t \in Y_n\}$ is then a realization of the associahedron K_{n+1} in the hyperplane $\sum_{i=1}^{n} x_i = 0$.

Example 5.5. For Y_3 we have 5 binary trees:



These correspond to the five vectors:

$$\begin{aligned} v_1 &= (1,2,3) \\ v_2 &= (2,1,3) \\ v_3 &= (1,4,1) \\ v_4 &= (3,1,2) \\ v_5 &= (3,2,1) \end{aligned}$$

The Loday associahedron is the convex hull of these points.

Another realization is due to Chapoton, Fomin and Zelevinsky, [4], and in fact provides a generalization of the associahedron through techniques in cluster algebras and semisimple Lie algebras. We sketch the formal definition here, and will return to it in greater detail.

Definition 5.6. Let Φ be a rank n finite root system,with the set of simple roots Π, and the set of positive roots $\Phi_{>0}$. Define $\Phi_{\geq -1} := \Phi_{>0} \cup (-\Pi)$. We also have a notion of compatible roots, see [11] for a precise definition. We can then define $\Delta(\Phi)$, a simplicial complex with elements of $\Phi_{\geq -1}$ as vertices and simplices are subsets of mutually compatible elements of $\Phi_{\geq -1}$. The maximal simplices are called **clusters**.

Definition 5.7. The **root system of type** A_n is the set $\Phi := \Phi(A_n) = \{e_i - e_j, 1 \leq i \neq j \leq n+1\} \subset \mathbb{R}^{n+1}$. where $\{e_i\}$ is the standard basis. The **simple** roots of type A_n are the elements of the set $\Pi = \{\alpha_i = e_i - e_{i+1} : 1 \leq i \leq n\}$. The set of **positive** roots is $\Phi_{>0} = \{e_i - e_j : i < j\}$, and the set of almost positive roots is $\Phi_{\geq -1} := \Phi_{>0} \cup (-\Pi)$.

Theorem 5.8 ([11]). *All clusters are of the same dimension i.e. $\Delta(\Phi)$ is pure. Moreover, the simplicial cones generated by the clusters form a complete simplicial fan in $Q_{\mathbb{R}}$, where Q is the root lattice.*

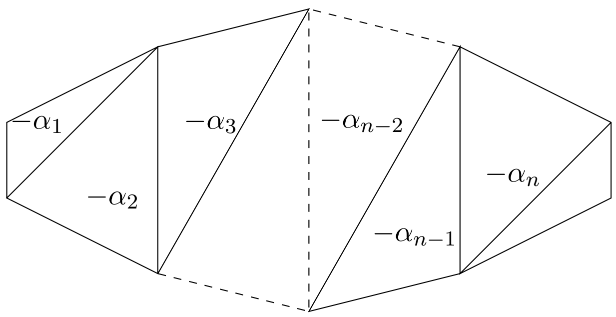
Here, a **complete** fan is one for which the union of its cones is the entire space.

Theorem 5.9 ([4]). *The simplicial fan defined by $\Delta(\Phi)$ is the normal fan of a simple convex polytope, called a generalized associahedron.*

In particular, for a root system of type A_n [15] $\Delta(\Phi)$ defines a realization of the standard associahedron.
Once we allow for generalized associahedra, we obtain infinitely many non-equivalent realizations, either by the construction of Hohlweg-Lange [14], Santos [3], or Gelfand-Kapranov-Zelevinsky [13]. Some of these are families we have seen before: the only realization obtained in both the construction of Hohlweg-Lange and Santos is the Chapoton-Fomin-Zelevinsky associahedron. Most, however, are beyond the scope of this discussion and have no bearing on our results.

We shall instead focus on the Chapoton-Fomin-Zelevinsky (CFZ) associahedron of type A_n for now, as Ceballos [1] provides us with a particularly nice description of the simplicial fan. We invite the reader to examine Section 9 to see figures of various realizations of the associahedron and generalized associahedra.

For the root system of type A_n, we can identify each of $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$ with a diagonal of the $(n+3)$-gon as shown:



We can then identify positive roots

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$$

with the unique diagonal crossing $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$. Our compatibility condition reduces to α, β are compatible if the corresponding diagonals do not cross. From the results of Theorem 5.8, this defines a complete simplicial fan in the space

$$\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 + \cdots + x_{n+1} = 0\}.$$

By Theorem 5.9, it is the normal fan of a polytope. In [4], it is shown to be the associahedron, and explicit inequalities are given to describe it.

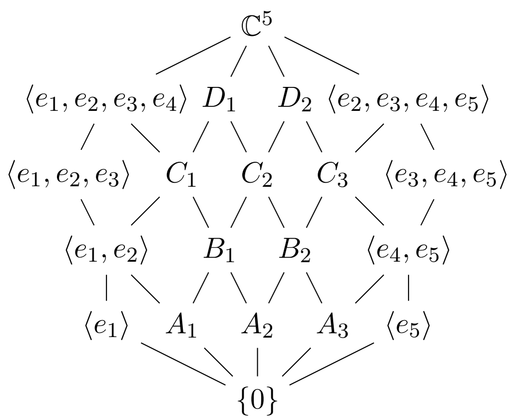
We can now discuss the gives goal of this project. We have seen that the Loday associahedron can be realized by a subword complex and the polytope of a toric variety. We will now provide the same realization for the CFZ associahedron. Specifically, we will show that it corresponds to the choice of

$$Q = (s_1\ s_3\ s_5\ \ldots\ s_{2k-1}\ s_2\ s_4\ \ldots\ s_{2k-2})^{k+1} \tag{1}$$

for $n = 2k$, and

$$Q = (s_1\ s_3\ s_5\ \ldots\ s_{2k-1}\ s_2\ s_4\ \ldots\ s_{2k})^{k+1}\ s_1\ s_3\ \ldots s_{2k-1} \tag{2}$$

for $n = 2k+1$ and to the variety described by the following sort of diagram ($n = 5$):

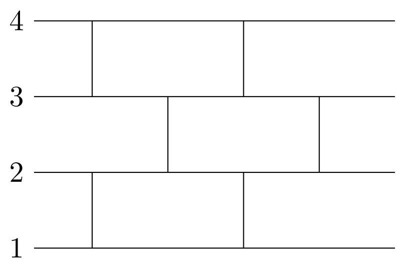


5.2 Sorting networks

By completeness of the fan, and convexity of the polytopes involved, to show that we obtain the CFZ associahedron, we need only to show that we obtain the root system of type A_n as our facet normals. To do this, we introduce the theory of sorting networks, first developed in [24]. We continue to work over the Coxeter group S_n with generators $S = \{s_1, \ldots, s_{n-1}\}$.

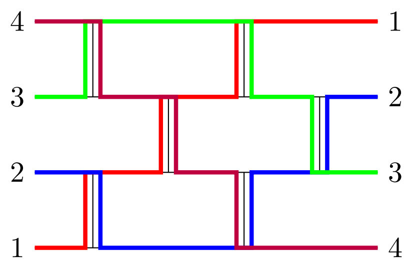
Definition 5.10. The **sorting network** $\mathcal{N}_Q$ of a word $Q = (q_1, q_2, \ldots q_m)$ in S is a diagram consisting of n horizontal lines, called **levels**, and m vertical segments, called **commutators**, drawn from left to right so that no two commutators share a common endpoint and if $q_k = s_i$, the k^{th} commutator connects levels i and $i+1$. A **brick** of $\mathcal{N}_Q$ is a connected component of the complement of $\mathcal{N}_Q$, bounded on the left by a commutator.

Example 5.11. Let $Q = (s_1, s_3, s_2, s_1, s_3, s_2)$. The sorting network, $\mathcal{N}_Q$ is then



Definition 5.12. A **pseudoline** supported by $\mathcal{N}_Q$ is a path on $\mathcal{N}_Q$, travelling monotonically left to right. A commutator of $\mathcal{N}_Q$ is called a **crossing** between two pseudolines if it is crossed by both pseudolines, and is otherwise called a **contact**. A **pseudoline arrangement** on $\mathcal{N}_Q$ is a collection of n pseudolines supported by $\mathcal{N}_Q$ such that each pair have at most one crossing and no other intersection.

Example 5.13. An example of a pseudoline arrangement on $\mathcal{N}_{(s_1,s_3,s_2,s_1,s_3,s_2)}$ is



One will note that a pseudoline arrangement on $\mathcal{N}_Q$ determines a permutation $w \in S_n$. In particular, it describes w_0, the longest permutation in S_n .

By considering pseudoline arrangements on $\mathcal{N}_Q$, we can define a simplicial complex $\Delta(\mathcal{N}_Q)$, known to be isomorphic to the subword complex $\Delta(Q, w_0)$. As such, we can define a **brick polytope** $B(\mathcal{N}_Q)$ of an irreducible sorting network, analogous to that for subword complexes. We direct the reader to [26] or [10] for a precise definition. We will, however, note the following fact, which is immediate from the definitions.

Fact 5.14. The brick polytope, in the subword complex sense, and the brick polytope, in the sorting network sense, are identical up to translation in space.

Thus we may use the theory of sorting networks to discuss the polytopes of Escobar's varieties. We note in particular the following result.

Theorem 5.15 (Corollary 3.21 [25])**.** *The facet normals of the brick polytope $B(\mathcal{N}_Q)$ for a pseudoline arrangement on irreducible $\mathcal{N}_Q$ the determines the permutation w are precisely all facet normal vectors of the incidence cones of the contact graphs of the pseudoline arrangements supported by $\mathcal{N}_Q$. Representatives for them are given by the characteristic vectors of the sinks of the minimal directed cuts of these contact graphs.*

The understanding of all aspects of this theorem is not vital to our discussion. We note only that this allows us to compute the facet normals of the brick polytope, as all Q we consider give rise to irreducible $\mathcal{N}_Q$, using [25, Remark 4.7] as follows.

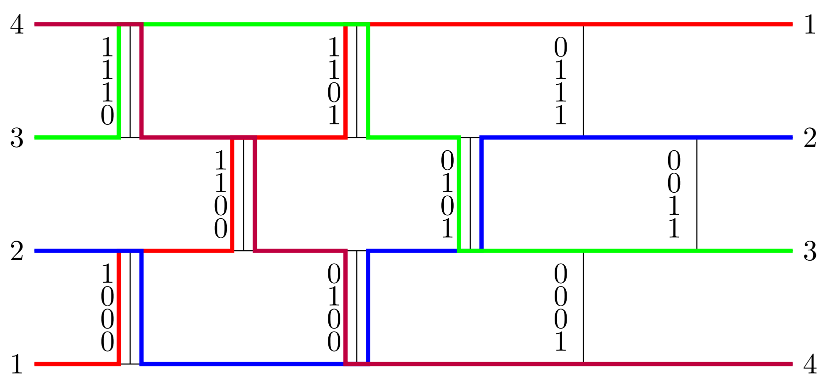
Definition 5.16. The **greedy pseudoline arrangement** supported by $\mathcal{N}_Q$ is the unique source of the graph of flips [24]. Equivalently, it is characterized by the properties the any of its contacts are to the right of the corresponding crossings, and that it determined the longest permutation w_0.

We then obtain a facet normal v_i for each commutator in $\mathcal{N}_Q$, with entries determined by:

$$j^{th} \text{ component of } v_i = \begin{cases} 0 & j^{th} \text{ pseuodline passes above commutator } i \\ 1 & j^{th} \text{ pseuodline passes below commutator } i \end{cases}$$

for the greedy pseudoline arrangement. We will illustrate this with a quick example.

Example 5.17. The facet normals of $B(\mathcal{N}_{(s_1,s_3,s_2,s_1,s_3,s_2,s_1,s_3,s_2)})$ are given by



5.3 The brick manifold of Chapoton-Fomin-Zelevinsky

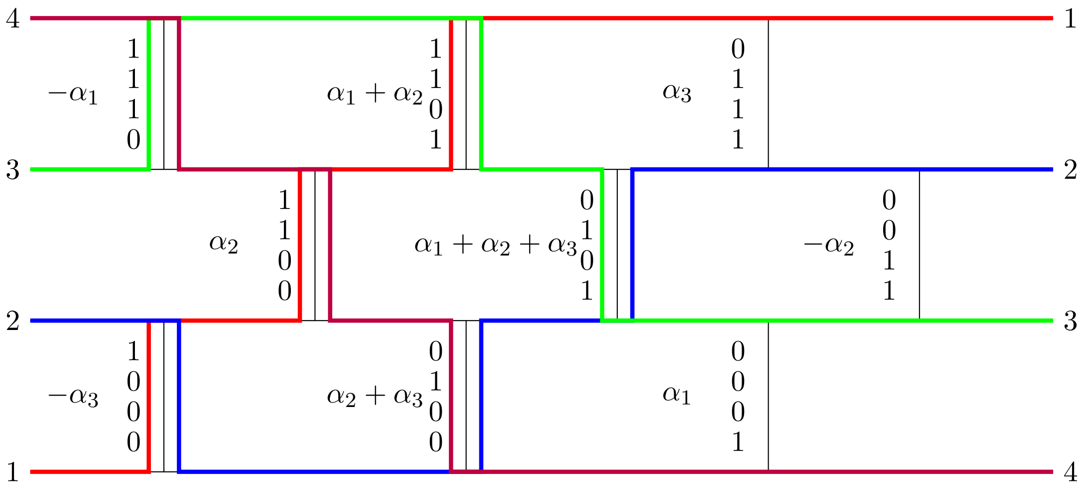
We can now show that the polytope of the varieties determined by Q as in Equations (1) and (2) is the CFZ associahedron. One must, however, note that due to our construction, we must consider our normal vectors equivalent up to a positive multiple of $\mathbb{I}$, the vector with 1 in every component.

Theorem 5.18. *$B(Q, w_0)$ for Q as in (1) or (2), as thus the polytope of the toric varieties $m_Q^{-1}(w_0B)$, is the Chapoton-Fomin-Zelevinsky associahedron. As such, we refer to $m_Q^{-1}(w_0B)$ as the CFZ manifold.*

Example 5.19. Consider $Q = (s_1, s_3, s_2, s_1, s_3, s_2, s_1, s_3, s_2)$. This gives the variety determined by the diagram

$$\begin{tikzcd}
 & & & \mathbb{C}^4 \arrow[dlll, dash] \arrow[dl, dash] \arrow[dr, dash] \arrow[drrr, dash] & & & \\
\langle e_1, e_2, e_3 \rangle \arrow[d, dash] & & C_1 \arrow[dll, dash] \arrow[d, dash] & & C_2 \arrow[dll, dash] \arrow[d, dash] & & \langle e_2, e_3, e_4 \rangle \arrow[dll, dash] \arrow[d, dash] \\
\langle e_1, e_2 \rangle \arrow[d, dash] \arrow[drr, dash] & & B_1 \arrow[d, dash] \arrow[drr, dash] & & B_2 \arrow[d, dash] \arrow[drr, dash] & & \langle e_3, e_4 \rangle \arrow[d, dash] \\
\langle e_1 \rangle \arrow[drrr, dash] & & A_1 \arrow[dr, dash] & & A_2 \arrow[dl, dash] & & \langle e_4 \rangle \arrow[dlll, dash] \\
 & & & \{0\} & & &
\end{tikzcd}$$

which has corresponding polytope determined by the normal vectors



where we consider addition modulo $\mathbb{I}$. Note that there are precisely the normal vectors of the CFZ associahedron of type A_3, up to a change of basis. Thus the polytope of the variety $m_Q^{-1}(w_0B)$, $B(Q, w_0)$ is affinely equivalent to the Chapoton-Fomin-Zelevinsky associahedron.

Proof. Our result holds in the case $n = 4$ and it can be easily verified for $n = 1, 2, 3$. We shall proceed from here by induction, showing that our facet normals satisfy the desired relations. Assume that we get desired normal fan from $n = k - 1$ and consider the greedy pseudoline arrangement for $n = k$. We label the normal vector attached to the leftmost commutator joining levels i and $i+1$ by $(-1)^i \alpha_i^k$. It is an obvious property of the greedy pseudoline arrangement that the normal vector attached to the rightmost commutator joining levels $k - i$ and $k - i + 1$ is then $(-1)^{i+1}\alpha_i^k$. Note that we then have that $\alpha_i^k = [0, \ldots, 0, 1, \ldots, 1]$ with i 0's if i is odd and $\alpha_i^k = [1, \ldots, 1, 0, \ldots, 0]$ with i 1's if i is even.We have two cases to discuss for the remaining vectors. We also require the following assumption.

If $v = \sum_{i \in I_v} \alpha_i^k$ is the normal vector attached the a commutator below the k-th pseudoline, I_v is an interval containing at least as many even integers as odd. If $|I_v|$ is even, the least element of I_v is odd and vice versa.

If v is above the k-pseudoline, I_v contains at least as many odd integers as even. If $|I_v|$ is even, the least element of I_v is even, and similarly if I_v is odd, so is it's least element.

k is even Clearly, below the $k-1$-th pseudoline, the greedy pseudoline arrangement is identical to that of the $k-1$ case and thus the vector associated to each commutator is identical to the corresponding vector in the $k-1$ case in the first $k-1$ components and a 0 in the k-th component. In fact α_i^k cannot differ from α_i^{k-1} in the first $k - 1$ entries, for any $1 \leq i < k$. Thus, if v^k is the vector corresponding to a commutator below the $k - 1$th pseudoline and the corresponding vector v^{k-1} had $v^{k-1} = \sum_{i \in I_v} \alpha_i^{k-1}$, for some interval $I_v \subset \{1, 2, \ldots, k - 1\}$, then $(v^k) = \sum_{i \in I_v} \alpha_i^k$ in the first $k - 1$ entries. As I_v contains at least as many even integers as odd integers, modulo $\mathbb{I}$, the k-th component of the sum can be taken to be 0. To see this, note that

$$\alpha_i^k + \alpha_{i+1}^k = e_{i+1} \text{ if } i \text{ is odd}$$

and we will only have such sums with $i < k - 1$. Thus every odd-even pair in I_v contributes 0 to the k-th component. If we have an unpaired element of I_v, it must be even, and so contributes 0 to the k-th component. Thus below the $(k - 1)$-th pseudoline, our facet normals are as needed, and our additional assumption holds. Next we consider the section of our diagram above the k-th pseudoline. Once again, the greedy pseudoline arrangement is identical to the greedy pseduoline arrangement of the $k - 1$ case above the $k - 1$-th pseudoline. Thus, the vector associated to each commutator is identical to the corresponding vector in the $k-1$ case in the first $k - 1$ components, but has a 1 in the k-th component. Next, note that

$$\alpha_i^k + \alpha_{i+1}^k = \mathbb{I} - e_{i+1} \text{ if } i \text{ is even}$$

We similarly have that, if v^k is the vector corresponding to a commutator below the $(k-1)$th pseudoline and the corresponding vector v^{k-1} had $v^{k-1} = \sum_{i \in I_v} \alpha_i^{k-1}$, for some interval $I_v \subset \{1, 2, \ldots, k-1\}$, then $(v^k) = \sum_{i \in I_v} \alpha_i^k$ in the first $k-1$ entries. As I_v contains at least as many odd elements as even, we can see that the k-th entry of $\sum_{i \in I_v} \alpha_i^k$ must be 1.
If $|I_v|$ is even, then

$$\begin{aligned}\sum_{i \in I_v} \alpha_i^k &= \sum_{i \in I_v,\ i \text{ odd}} \alpha_{i-1}^k + \alpha_i^k \\ &= \sum_{i \in I_v,\ i \text{ odd}} \mathbb{I} - e_i \\ &= \sum_{i \in I_v,\ i \text{ even or } i \notin I_v} e_i\end{aligned}$$

which contains e_k so the k-th entry is 1. If $|I_v|$ is odd, then let $s = \max\{I_v\}$. s must be odd and so

$$\begin{aligned}\sum_{i \in I_v} \alpha_i^k &= \alpha_s^k + \sum_{i \in I_v,\ i \text{ even}} \alpha_{i-1}^k + \alpha_i^k \\ &= \alpha_s^k \sum_{i \in I_v,\ i \text{ even}} e_i\end{aligned}$$

As before, the sum over $i \in I_v$, i even, will have 0 k-th component, so the overall sum will have 1 as it's k-th component. Thus, above the k-th pseudoline, we get the desired relations among our facet normal vectors, and our inductive assumptions still hold.

Finally we consider v associated to commutators between the $(k-1)$-th and k-th pseudolines. We know the bottom right vector is α_{k-1}^k and the top left vector is $-\alpha_{k_1}^k$. Now consider v^n, the vector associated to commutator joining the j-th and $(j+1)$-th levels. This will agree in the first $k-1$ components with the vector v^{k-1} associated the the commutator connecting the $(j-1)$-th and j-th traversed by the $(k-1)$-th pseudoline in the $k-1$ case, but will have a 1 in the final component. Thus, if $v^{k-1} = \sum_{i \in I_v} \alpha_i^{k-1}$, we have that, in the first $k-1$ components, $v^k = \sum_{i \in I_v} \alpha_i^k$. By our inductive assumptions, $\sum_{i \in I_v} \alpha_i^k$ has a 0 in the k-th component, and so $\alpha_{k-1}^k + \sum_{i \in I_v} \alpha_i^k$ has a 1 in the k-th component. Thus $v^k = \sum_{i \in I_v \cup \{k-1\}} \alpha_i^k$. We get the desired relations, and it is easy to check that our inductive assumptions still hold.

k is odd The proof is nearly identical to the previous case, but we divide our arrangement into the area *below* the k-th pseudoline, the area *above* the $(k-1)$-th pseudoline, and the area between.

Hence we obtain the desired normal fan in every case, up to affine transformation, and, therefore, the Chapoton-Fomin-Zelevinksy associahedron. □

6 Non-symmetric operads and associahedra

6.1 Non-symmetric operads

In [8] Dotsenko, Shadrin and Vallette, while investigating noncommutative analogues of the Deligne-Mumford compactification of moduli spaces of genus zero curves with marked points, find that these analogues can be interpreted as Escobar's brick manifolds and, as a result of this, are able to impose the structure of an operad on these varieties. We find that, while we have a similar topological structure in space of CFZ manifolds,we cannot impose the same operad structure. In Section 7, we will provide a partial explanation as to why. First we shall provide a brief background in the theory of operads. As we cannot possibly give a comprehensive description, we suggest a few texts for the interested reader: [7], [21], [12].

Definition 6.1. A **non-symmetric operad**, shortened to ns-operad, is an algebraic structure $\mathcal{P}$ consisting of

- A sequence $\{P(n)\}_{n\in\mathbb{N}}$ of sets, whose elements are called n-ary operations.
- An element $1 \in P(1)$, called the identity.
- For all $n, k_1, \ldots, k_n \in \mathbb{N}$, a composition function

$$\begin{aligned}\circ : P(n) \times P(k_1) \times \cdots \times P(k_n) &\to P(k_1 + \ldots + k_n)\\ (f, f_1, f_2, \ldots, f_n) &\mapsto f \circ (f_1, \ldots, f_n)\end{aligned}$$

  satisfying the following axioms

  - **Identity**:
  $$f \circ (1, \ldots 1) = f = 1 \circ f$$
  - **Associativity**:
  $$\begin{aligned}&f \circ (f_1 \circ (f_{1,1}, \ldots, f_{1,k_1}), \ldots, f_n \circ (f_{n,1}, \ldots, f_{n,k_n}))\\ &= (f \circ (f_1, \ldots, f_n)) \circ (f_{1,1}, \ldots, f_{1,k_1}, \ldots, f_{n,1}, \ldots, f_{n,k_n})\end{aligned}$$

An equivalent, and perhaps easier, definition is the following.

Definition 6.2. An ns-operad $\mathcal{P}$ is a collection $\{P(n)\}_{n=1}^{\infty}$, identity element $1 \in P(1)$, and an **infinitesimal composition at slot** i, a map

$$\begin{aligned}\circ_i : P(m) \times P(n) &\to P(m + n - 1)\\ \circ_i : f \otimes g &\mapsto f \circ_i g\end{aligned}$$

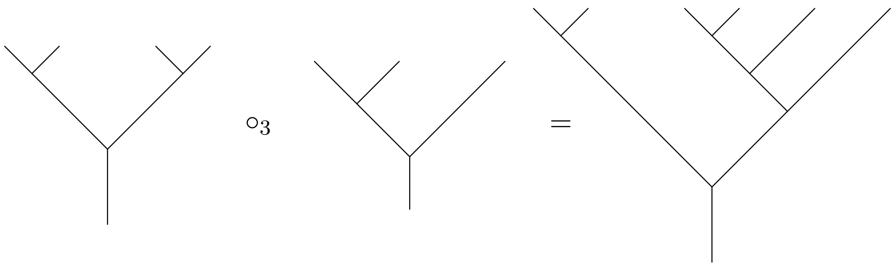
such that for $f \in P(n)$, $g \in P(m)$, $h \in P(r)$:

- $f \circ_i (g \circ_j h) = (f \circ_i g) \circ_{i+j-1} h$ for $i \leq j \leq i+m-1$.
- $(f \circ_i g) \circ_j h = \begin{cases} (f \circ_{j-m+1} h) \circ_i g, \text{ if } i+m \leq j \leq n+m-1 \\ (f \circ jh) \circ i+r-1g, \text{ if } 1 \leq j \leq i-1 \end{cases}$
- $f \circ_i 1 = f = 1 \circ_1 f$.

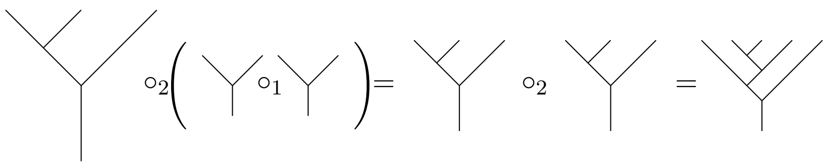
Example 6.3. Perhaps the easiest examples of an ns-operad is the operad of trees. We take

$$P(n) = \{\text{Rooted trees with } n \text{ leaves, modulo nodes of degree 2}\}.$$

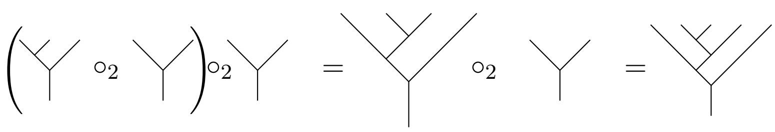
Then The only element of $P(1)$ is identity element 1 consisting of a single edge. Labeling the leaves $1, 2, \ldots, n$ of a tree in $P(n)$, we define infinitesimal compostion $T_1 \circ_i T_2$ to be the tree obtained by grafting the root of T_2 to leaf i of T_1.



It is easy to see that the all axioms are satisfied. Composition with the identity adds a vertex of degree two, which we neglect. The j-th leaf of T_2 corresponds to the $(i+j-1)$-th leaf of $T_1 \circ_i T_2$. For example



and



This example will prove useful in later discussion, as it will provide a stratification of our variety. This should, however, not be surprising, as an alternative description of the Tamari lattice can be made in terms of binary trees and we have established strong connections between Escobar's varieties and the associahedron.

6.2 An operad structure on brick manifolds

We shall now describe the operad structure on those brick manifolds corresponding to the Loday associahedron, which we shall refer to as the Loday manifold, to distinguish it from the CFZ manifold. To be more consistent with [8], we shall also introduce slightly different notation.

Definition 6.4. Denote by $[m, n]$ the set $\{m, m+1, \ldots, n\}$ for every $m \leq n \in \mathbb{N}$. To each such interval, we associate the vector space $G(m, n) := \mathrm{Span}(e_m, e_{m+1}, \ldots, e_{n-1}) \subset \mathbb{C}^N$, for some large $N > n$. We say $G(m, m) := \{0\}$.

Note that if $[m_1, n_1] \subset [m_2, n_2]$, we have a natural inclusion $G(m_1, n_1) \subset G(m_2, n_2)$. Also, note ease of notation, if we let $I := [m, n]$, we shall write $G(I)$ for $G(m, n)$.

Definition 6.5. Points of the **brick manifold** $\mathcal{B}(I)$, where $I = [m, n]$, are collections of subspaces $V_{i,j} \subset G(I)$, for all proper intervals $[i, j] \subsetneq [1, n]$ that satisfy the following:

- $\mathrm{dim} V_{i,j} = j - i + 1$
- $V_{i,j} \subset V_{i-1,j}$ for all $i > m$
- $V_{i,j} \subset V_{i,j+1}$ for all $j < n$
- $V_{m,j} = G(m, j+1)$
- $V_{i,n} = G(i-1, n)$

We call $\mathcal{B}([1, n])$ the **Loday manifold** and denote it by $\mathcal{B}(n)$.

It is quite easy to see that $\mathcal{B}(n)$ describe the same variety as Escobar's Loday manifold, $m_Q^{-1}(w_0 B)$.

Example 6.6. $\mathcal{B}(4)$ corresponds to $m_Q^{-1}(w_0 B)$, for $Q = (s_1, s_2, s_1, s_2, s_1)$, but is presented as

$$\begin{tikzcd}
 & & G(1,4) & & G(1,4) & & & & \cdots \mathrm{dim}3 \\
 & G(1,3) \arrow[ur] & & V_{2,3} \arrow[ul] \arrow[ur] & & G(2,4) \arrow[ul] & & & \cdots \mathrm{dim}2 \\
G(1,2) \arrow[ur] & & V_{2,2} \arrow[ul] \arrow[ur] & & V_{3,3} \arrow[ul] \arrow[ur] & & G(3,4) \arrow[ul] & & \cdots \mathrm{dim}1
\end{tikzcd}$$

Definition 6.7. For $i \in [m_1, n_1]$, we define $[m_1, n_1] \sqcup_i [m_2, n_2]$ to be the disjoint union $([m_1, n_1] \setminus \{i\}) \sqcup [m_2, n_2]$, identified with a copy of $[m_1, n_1 + n_2 - m_2]$ as follows:

$$\begin{aligned}
a \in [m_1, n_1] \setminus \{i\} &\leftrightarrow a \text{ if } a < i \\
a \in [m_1, n_1] \setminus \{i\} &\leftrightarrow a + n_2 - m_2 \text{ if } a > i \\
b \in [m_2, n_2] &\leftrightarrow b + i - 1
\end{aligned}$$

We are effectively replacing the element $i \in [m_1, n-1]$ with a copy of $[m_1, n_2]$ while preserving internal order.

Note that this correspondence establishes a bijection between basis elements of $G(I) \oplus G(J)$ and of $G(I \sqcup_i J)$ for every $i \in I$, where I, J are intervals. This extends to a family of linear bijections on the whole spaces, which we denote by $f^J_{I,i}$. We can now define an infinitesimal composition on $\{\mathcal{B}(n)\}_{n\in\mathbb{N}}$.

Theorem 6.8 ([8] Prop. 19)**.** *The maps $\circ_i$ make the collection $\{\mathcal{B}(n)\}_{n\in\mathbb{N}}$ into an ns-operad, where $\circ_i$ is defined by:*

$$\circ^J_{I,i} : \mathcal{B}(I) \times \mathcal{B}(J) \to \mathcal{B}(I \sqcup_i J)$$

considering I, J as disjoint sets and putting

$$\left(\{V^1_{i_1,j_1}\} \circ^J_{I,i} \{V^2_{i_2,j_2}\}\right)_{a,b} := \begin{cases} f^J_{I,i}(V^1_{a,b}), \textit{ for } a, b \in I,\ a \le b < i \\ f^J_{I,i}(V^2_{a,b}), \textit{ for } a \le b \in J,\ (a,b) \neq (\min(J), \max(J)) \\ f^J_{I,i}(V^1_{a,b}), \textit{ for } a, b \in I,\ i < a \le b \\ f^J_{I,i}(V^1_{a,i-1}) \oplus G(\min(J), b+1), \textit{ for } a \in I,\ a < i,\ b \in J,\ b < \max(J) \\ f^J_{I,i}(V^1_{i-1,b}) \oplus G(a-1, \max(J)), \textit{ for } a \in J,\ a > \min(J),\ b \in I,\ b > i \\ f^J_{I,i}(V^1_{i,i}) \oplus G(J), \textit{ for } (a,b) = (\min(J), \max(J)) \\ f^J_{I,i}(V^1_{a,i}) \oplus G(J), \textit{ for } a \in I,\ a < i,\ b = \max(J) \\ f^J_{I,i}(V^1_{i,b}) \oplus G(J), \textit{ for } b \in I,\ b > i,\ a = \min(J) \\ f^J_{I,i}(V^1_{a,b}) \oplus G(J), \textit{ for } a, b \in I,\ a < i < b \end{cases}$$

While the definition of this composition appears quite complicated, it is actually an extremely natural approach, which we believe the reader will see after trying a few examples themselves.

Example 6.9. The example of operad composition $\circ^J_{I,3} : \mathcal{B}(5) \times \mathcal{B}(5) \to \mathcal{B}(9)$ is shown below. For sake of clarity, we shall not include arrows indicating inclusion and we shall denote $f^J_{I,3}$ simply by f.

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & G(1,9) & & G(1,9) & & & & & & \\
 & & & & & & & G(1,8) & & f(V_{2,4}^1)\oplus G(3,7) & & G(2,9) & & & & & \\
 & & & & & & G(1,7) & & f(V_{2,3}^1)\oplus G(3,7) & & f(V_{3,4}^1)\oplus G(3,7) & & G(3,9) & & & & \\
 & & & & & G(1,6) & & f(V_{2,2}^1)\oplus G(3,7) & & f(V_{3,3}^1)\oplus G(3,7) & & f(V_{4,4}^1)\oplus G(3,7) & & G(4,9) & & & \\
 & & & & G(1,5) & & f(V_{2,2}^1)\oplus G(3,6) & & G(3,7) & & G(3,7) & & f(V_{4,4}^1)\oplus G(4,7) & & G(5,9) & & \\
 & & & G(1,4) & & f(V_{2,2}^1)\oplus G(3,5) & & G(3,6) & & f(V_{2,4}^2) & & G(4,7) & & f(V_{4,4}^1)\oplus G(5,7) & & G(6,9) & \\
 & & G(1,3) & & f(V_{2,2}^1)\oplus G(3,4) & & G(3,5) & & f(V_{2,3}^2) & & f(V_{3,4}^2) & & G(5,7) & & f(V_{4,4}^1)\oplus G(6,7) & & G(7,9) \\
 & G(1,2) & & f(V_{2,2}^1) & & G(3,4) & & f(V_{2,2}^2) & & f(V_{3,3}^2) & & f(V_{4,4}^2) & & G(6,7) & & f(V_{4,4}^1) & & G(8,9)
\end{array}
$$

These varieties can be stratified so that, infinitesimal composition is a bijection onto the closure of a particular stratum, allowing for another ways of finding the operad structure. We shall describe the stratification as follows.

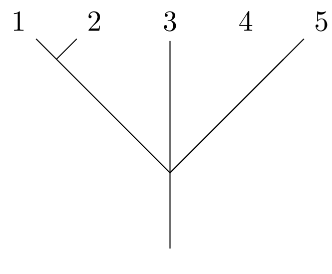
Definition 6.10. Let T be a planar rooted tree with n leaves such that all vertices have degree at least 3 and let $\mathcal{T}(n)$ be the set of all such trees. We label the leaves $1, \ldots, n$ from left to right. For any vertex, we call the incident edges farthest from the root its input edges. For a given edge e of the tree, denote by L_e the set of leaves of the subtree with root e.
The stratum $\mathcal{B}(n, T)$ of $\mathcal{B}(n)$ consists of all collections $\{V_{i,j}\}$ in $\mathcal{B}(n)$ satisfying the following conditions:

- For each edge e of T that is not a leaf, so that $L_e = \{l, l+1, \ldots, r\}$ for some $l < r$, we require that $V_{l,r-1} = V_{l+1,r} = G(l, r)$.
- For each edge e of T that is neither the root, nor the leftmost or rightmost input edge of a vertex, we require that $V_{l,r}$ is neither of the two possible $r - l + 1$-dimensional coordinates subspaces $G(l-1, r)$ or $G(l, r+1)$ of $G(l-1, r+1)$.

Theorem 6.11 ([8], Prop. 21)**.** *The subvarieties $\mathcal{B}(n, T)$, $T \in \mathcal{T}(n)$, form a stratification of $\mathcal{B}(n)$. To be exact:*

- *Each subvariety $\mathcal{B}(n, T) \subset \mathcal{B}(n)$ is isomorphic to*
$$\mathcal{B}(n, T) \simeq (\mathbb{C}^*)^{n-2-n_e}$$
where $n_e = n_e(T)$ is the number of internal edges of T, that is, edges which are neither leaves nor the root.
- *We have*
$$\mathcal{B}(n) = \sqcup_{T \in \mathcal{T}(n)} \mathcal{B}(n, T)$$
- *For any $T \in \mathcal{T}(n)$, the closure of $\mathcal{B}(n, T)$ in $\mathcal{B}(n)$ is the union of the subvarieties $\mathcal{B}(n, T')$, where T' is a planar tree from which T can be obtained by contracting some of the internal edges.*

Example 6.12. Consider the tree in $\mathcal{T}(n)$ as shown



The corresponding stratum in $\mathcal{B}(5)$ is given by collections $\{V_{i,j}\}$ such that

- $V_{1,1} = V_{2,2} = G(1,2)$
- $V_{4,4} = V_{5,5} = G(4,5)$
- $G(2,3) \neq V_{3,3} \neq G(3,4)$

Equivalently, the stratum is given by collections $\{V_{i,j}\}$ such that the following diagram holds:

$$\begin{tikzcd}[column sep=small]
 & & & G(1,5) & & G(1,5) & & & \\
 & & G(1,4) \arrow[ur] & & V_{2,4} \arrow[ul] \arrow[ur] & & G(2,5) \arrow[ul] & & \\
 & G(1,3) \arrow[ur] & \neq & V_{2,3} \arrow[ul] \arrow[ur] & & V_{3,4} \arrow[ul] \arrow[ur] & \neq & G(3,5) \arrow[ul] & \\
G(1,2) \arrow[ur] & & G(1,2) \arrow[ul] \arrow[ur] & & V_{3,3} \arrow[ul] \arrow[ur] & & G(4,5) \arrow[ul] \arrow[ur] & & G(4,5) \arrow[ul]
\end{tikzcd}$$

One might also note that the strata corresponding to binary trees consist of single points, and so each 0-dimensional stratum can be identified with a vertex of the associahedron. Indeed, each k-dimensional stratum can be identified with a k-dimensional face of the associahedron, by identifying each face with the unique tree T that can be obtained by contraction internal edges in the binary trees associated to its vertices [23, Section 5.8]. Moreover, one can easily see that infinitesimal composition $\circ_{I,i}^{J} : \mathcal{B}(I, T_1) \times \mathcal{B}(J, T_2) \to \mathcal{B}(I \sqcup_i J, T_1 \circ_i T_2$. That it, infinitesimal composition takes the product of strata corresponding to $T_1,\ T_2$, to the stratum corresponding to the composition of $T_1,\ T_2$.

6.3 Extensions of this operad

One would hope that we could define a similar natural operad structure on the CFZ manifolds. However, due to the nature of the inclusions diagram that must be satisfied, there is no obvious map. In order to attempt to define an infinitesimal composition, we shall first stratify the CFZ manifold and proceed from there.

Definition 6.13. We define points of the **Chapoton-Fomin-Zelevinksy manifold**, denoted $\mathcal{B}_{CFZ}(n)$ as collections $\{V_{i,j}\}$ satisfying the inclusion diagram as defined by $m_Q^{-1}(w_0 B)$ where Q is a word in the generators $\{s_1, \ldots, s_{n-2}\}$, as in 1 or 2, with labelling of subspaces as illustrated below.

As the labelling is somewhat counter intuitive and awkward to describe in general, we shall define it by showing the diagram for $n = 5$ and $n = 6$.

Example 6.14. $\mathcal{B}_{CFZ}(5)$ consists of collections $\{V_{i,j}\}$ satisfying the following diagram

$$
\begin{tikzcd}
 & G(1,5) & & G(1,5) & & G(1,5) & & \\
V_{4,4} = G(1,4) \arrow[ur] & & V_{1,3} = V_{5,5} \arrow[ul] \arrow[ur] & & V_{2,4} \arrow[ul] \arrow[ur] & & V_{3,5} = G(2,5) \arrow[ul] & \\
 & V_{1,2} = G(1,3) = V_{4,5} \arrow[ul] \arrow[ur] & & V_{2,3} \arrow[ul] \arrow[ur] & & V_{3,4} \arrow[ul] \arrow[ur] & & V_{4,5} = G(3,5) \arrow[ul] \\
V_{1,1} = G(1,2) \arrow[ur] & & V_{2,2} \arrow[ul] \arrow[ur] & & V_{3,3} \arrow[ul] \arrow[ur] & & V_{4,4} = G(4,5) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

The reader will note that we impose

$$V_{k,n} = V_{1,k-2}$$

and that we define the boundary subspaces twice. The first of these causes no issue, but simplifies discussion in later proofs. The second may seem alarming, but again, serves only to simplify notation. We need not panic, as these twice-defined subspaces are always fixed and are complementary in $G(1,n)$.

For $\mathcal{B}_{CFZ}(6)$, we obtain the following diagram. Once again, the reader will note that the boundary subspaces are defined twice, but also that, to the right of the flag consisting of $\{V_{1,k}\}_{k=1}^{n-2}$, we have

$$V_{i,j} \subset V_{r,s} \Leftrightarrow [i,j] \subset [r,s]$$

while to the left we have

$$V_{i,j} \subset V_{r,s} \Leftrightarrow [i,j] \supset [r,s]$$

$$
\begin{tikzcd}
 & & G(1,6) & & G(1,6) & & G(1,6) & & \\
 & V_{5,5}=G(1,5) \arrow[ur] & & V_{1,4}=V_{6,6} \arrow[ul] \arrow[ur] & & V_{2,5} \arrow[ul] \arrow[ur] & & V_{3,5}=G(2,6) \arrow[ul] & \\
V_{4,5}=G(1,4) \arrow[ur] & & V_{1,3}=V_{5,6} \arrow[ul] \arrow[ur] & & V_{2,4} \arrow[ul] \arrow[ur] & & V_{3,5} \arrow[ul] \arrow[ur] & & V_{4,6}=G(3,6) \arrow[ul] \\
 & V_{1,2}=G(1,3)=V_{4,6} \arrow[ul] \arrow[ur] & & V_{2,3} \arrow[ul] \arrow[ur] & & V_{3,4} \arrow[ul] \arrow[ur] & & V_{4,5}=G(4,6) \arrow[ul] \arrow[ur] & \\
V_{1,1}=G(1,2) \arrow[ur] & & V_{2,2} \arrow[ul] \arrow[ur] & & V_{3,3} \arrow[ul] \arrow[ur] & & V_{4,4} \arrow[ul] \arrow[ur] & & V_{5,5}=G(5,6) \arrow[ul]
\end{tikzcd}
$$

We note also that the subspaces $V_{i,i}$ are 1 dimensional for $i < \lfloor \frac{n}{2} \rfloor + 2$ and $(n-2)$-dimensional for $i > \lfloor \frac{n}{2} \rfloor + 2$. Combining this with our prior observation regarding subspace inclusion, one can easily label all subspaces in such a diagram, simply by labelling the 1-dimensional subspaces $V_{1,1}, V_{2,2}, \ldots$ from left to right and then "wrapping around" into the $(n-2)$-dimensional subspaces. As such, this is how we shall define our labelling. We shall refer to the area to the left of $\{V_{1,k}\}$ as the **wrap** and will signify that $V_{i,j}$ is in the wrap with a superscript w, $V^w_{i,j}$, in cases of ambiguity.

Due to the shape of our inclusion diagram, we can say significantly less about subspaces in $\mathcal{B}_{CFZ}(n)$ than we could about those in $\mathcal{B}(n)$. For example, while in $\mathcal{B}(n)$ $V_{i,j}$ was a codimension 1 subspace of $G(i-1, j+1)$, we cannot say anything quite as specific. We summarize what properties we can find in the following result.

Proposition 6.15. *The collections of subspaces $\{V_{i,j}\}$ that make up $\mathcal{B}_{CFZ}(n)$ satisfy the following:*

- *$V_{i,j}$ is in the wrap if $j \geq \lfloor \frac{n}{2} \rfloor + 2$ and $i + j \geq \beta_n := \min(4+2\lfloor \frac{n}{2} \rfloor, 4+2\lfloor \frac{n}{2} \rfloor + (-1)^{n+1})$.*
- *If $i + j \in \{4 + 2\lfloor \frac{n}{2} \rfloor, 4 + 2\lfloor \frac{n}{2} \rfloor + (-1)^{n+1}\}$, then $V_{i,j}$ is a twice defined boundary set and $V^w_{i,j} = G(1,n) \setminus V_{i,j}$.*
- *Outside the wrap $V_{i,j} \subset V_{r,s} \Leftrightarrow [i,j] \subset [r,s]$.*
- *Inside the wrap $V^w_{i,j} \subset V^w_{r,s} \Leftrightarrow [i,j] \supset [r,s]$.*
- *$V^w_{k,n} = V_{1,k-2}$ and so $V^w_{k,k} \supset V_{1,k-2}$.*
- *Outside the wrap, $V_{i,j} \subset G(\max(1, 2i-4), \min(2j-1, n))$.*
- *Outside the wrap, with $j \geq \lfloor \frac{n}{2} \rfloor + 2$, $V_{i,j} \supset G(2n+2-2j, n)$.*

- *Inside the wrap,* $V^w_{i,j} \supset G(1, 2n-2j+3)$*.*

Proof. All properties follow immediately the definition of our labelling. We invite the reader to work through the details themselves. $\square$

We can now define our stratification. Our description is less elegant than that of [8], and as such, the proof slightly less efficient.

Definition 6.16. Given a planar rooted tree $T \in \mathcal{T}(n)$ as before, we define $\mathcal{B}_{CFZ}(n,T) \subset \mathcal{B}_{CFZ}(n)$ to be the set of all collections $\{V_{i,j}\} \in \mathcal{B}_{CFZ}(n)$ satisfying:

- For any edge e that is not a leaf of T, let $L_e = \{l, l+1, \ldots, r\}$ be the set of leaves of the subtree with root e. We then require that $V_{l,r-1} = V_{l+1,r}$.
- If there does not exist e an edge of T for which $L_e = \{l, \ldots, r\}$, we then require that $V_{l,r-1} \neq V_{l+1,r}$.

The reader will note that this is a significantly cruder variation of Definition 6.10, but when applied to $\mathcal{B}(n)$, becomes equivalent.

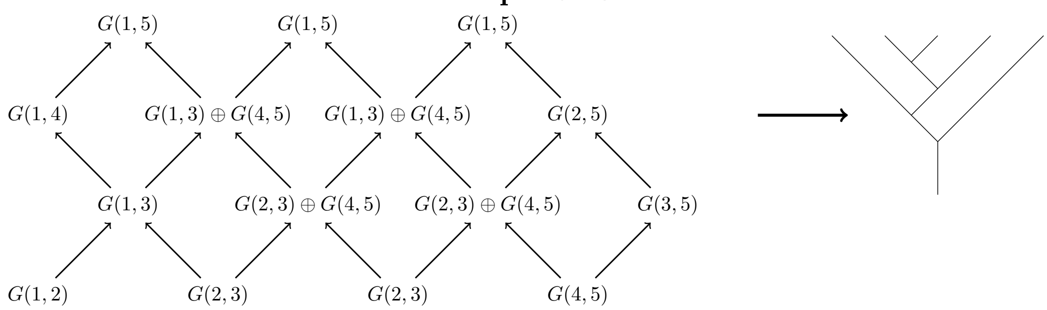
Theorem 6.17. $\mathcal{B}_{CFZ}(n)$ *is a disjoint union of of the subvarieties* $\mathcal{B}_{CFZ}(n,T)$*.*

$$\mathcal{B}_{CFZ}(n) = \sqcup_{T \in \mathcal{T}(n)} \mathcal{B}_{CFZ}(n,T)$$

Proof. Clearly, $\mathcal{B}_{CFZ}(n,T_1) \cap \mathcal{B}_{CFZ}(n,T_2) = \emptyset$ if $T_1 \neq T_2$, as the equalities in the collections $\{V_{i,j}\}$ uniquely determine it's stratum. Thus we need only show that, if $\{V_{i,j}\} \in \mathcal{B}_{CFZ}n$, there exists $T \in \mathcal{T}(n)$ such that $\{V_{i,j}\} \in \mathcal{B}_{CFZ}(n,T)$.

Clearly, given a collection, we can attempt to construct a tree T. Start with a collection of vertices $\{1, \ldots, n\}$. Call this Tier 0. For every equality $V_{i,j-1} = V_{i+1,j}$, we create a Tier $j-i$ vertex v with an edge connecting i and v, and an edge connecting j and v. We also create a Tier ∞ vertex, from which we draw an edge to the vertices 1 and n. Then for every vertex v, with edges going to i, j, we draw an edge to the next lowest Tier vertex v' with edges going to $k \leq i$, $l \geq j$. If $k = i$, delete the edge between v' and i and similarly if $j = l$. If there is a Tier 0 vertex v_0 with no incident edges, draw an edge to the lowest Tier vertex with edges going to $i < v_0 < j$.

Example 6.18.



Clearly, this construction returns T is performed on any point of $\mathcal{B}_{CFZ}(n,T)$, so in order to establish our result, we simply need to show that this construction always returns an element of $\mathcal{T}(n)$. If for some collection $\{V_{i,j}\}$ it doesn't, that implies there exist $i < j \leq k < l$ for which

$$V_{i,k-1} = V_{i+1,k}$$
$$V_{j,l-1} = V_{j+1,l}$$

We will examine this in cases. We shall only describe one in detail, as the others while progressing distinctly, are similarly proven.

1. Suppose $\ell < \lfloor \frac{n}{2} \rfloor + 2$. Then we have that none of $V_{i,k-1},\ V_{i+1,k},\ V_{j,l-1}, V_{j+1,l}$ are in the wrap and so

$$V_{j,k} \subset V_{i+1,k} = V_{i,k-1} \subset G(\max(2i-2,1), 2k-3)$$
$$V_{j,k} \subset V_{j,l-1} = V_{j+1,l} \subset G(2j-2, \min(2l-3,n))$$

using results from Proposition 6.15. Then, as we must have $2j-2 \geq 2$ and $2k-3 < n$, we conclude

$$V_{j,k} \subset G(2j-2, 2k-3).$$

But we also have that $V_{i,j-1} \subset V_{i,k-1} = V_{i+1,k}$ and so $V_{i+1,k} \sup V_{i,j-1} \cup V_{j,k}$ and

$$V_{i,j-1} \subset V_{1,j} \subset G(1, 2j-3).$$

So $V_{i,j-1} \cap V_{j,k} \subset G(1,2j-3) \cap G(2j-2,2k-3) = \emptyset$. Therefore

$$V_{i+1,k} \supset V_{i,j-1} \oplus V_{j,k}$$
$$\Leftrightarrow \dim V_{i+1,k} \geq \dim V_{i,j-1} + \dim V_{j,k}$$
$$\Leftrightarrow k-i \geq k-i+1$$

a clear contradiction. Hence, this case cannot occur.

2. If $l \geq \lfloor \frac{n}{2} \rfloor + 2$, but $k + l < \beta_n$, we obtain a contradiction by similar analysis.
3. If $l \geq \lfloor \frac{n}{2} \rfloor + 2$, $k \geq \frac{1}{2}\beta_n$, but $j + l < \beta_n$, we again obtain a contradiction.
4. If $l \geq \lfloor \frac{n}{2} \rfloor + 2$, $k \geq \frac{1}{2}\beta_n$, $j = k$ and $i + k < \beta_n$, we similarly obtain a contradiction
5. If $l \geq \lfloor \frac{n}{2} \rfloor + 2$, $k \geq \frac{1}{2}\beta_n$, $j + l \geq \beta_n$ and $i + k \geq \beta_n$, we again obtain a contradiction.

Thus, every point in $\mathcal{B}_{CFZ}(n)$ is contained in one of the strata and our result is proven. □

Theorem 6.19. *For any $T \in \mathcal{T}(n)$, the closure of $\mathcal{B}_{CFZ}(n,T)$ in $\mathcal{B}_{CFZ}(n)$ is the union of the subvarieties $\mathcal{B}_{CFZ}(n,T')$, where T' is a planar tree from which T can be obtained by contracting some of the internal edges.*

Proof. Points in the closure of $\mathcal{B}_{CFZ}(n,T)$ consist of collections $\{V_{i,j}\}$ satisfying the equality constraints as determined by T, but allowing the possibility that two subspaces be equal, even if it is not required by T. From our construction, the only way the corresponding tree can differ from T is by introducing additional vertices and changing edges which were not leftmost or rightmost input edges into leftmost or rightmost input edges. From here, it is clear that T can then be obtained by contracting edges between these additional vertices and the vertices in T. Thus the claim is apparent. $\square$

Theorem 6.20. *Each subvariety $\mathcal{B}_{CFZ}(n,T) \subset \mathcal{B}_{CFZ}(n)$ is isomorphic to*

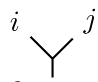
$$\mathcal{B}_{CFZ}(n,T) \simeq (\mathbb{C}^*)^{n-2-n_e}$$

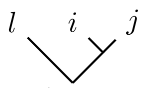
where $n_e = n_e(T)$ is the number of internal edges of T, that is, edges which are neither leaves nor the root. Equivalently,

$$\mathcal{B}_{CFZ}(n,T) \simeq (\mathbb{C}^*)^{i_e}$$

*where $i_e = i_e(T)$ is the number of edges of T that are either leftmost nor rightmost input edges of a vertex, which we shall refer to as **inner** edges of T.*

Proof. It is reasonably apparent than any inner edge corresponds to a degree of freedom in the collection $\{V_{i,j}\} \subset \mathcal{B}_{CFZ}(n,T)$ and so $\dim(\mathcal{B}_{CFZ}(n,T)) \geq i_e$. We will now show that every degree of freedom corresponds to an inner edge. Once again, we must split into cases with similar proofs, and as such, we shall only prove the first in detail. For any subspace $V_{k,l}$ which can vary while remaining in $\mathcal{B}_{CFZ}(n,T)$, this "variation" must appear in all subspaces containing $V_{k,l}$. Hence, there is subspace of minimal dimension containing this "variation", which we shall denote $V_{i,j}$.

1. If $V_{i,j}$ is not in the wrap and $i \neq j$, then our tree must contain 

where the root of this subtree must be an inner edge, or a leftmost input edge or a rightmost input edge. Suppose it is a rightmost input edge. Then there exists a maximal $1 \leq l < i$ such that we have 

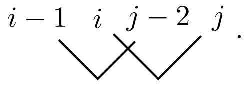
as a subtree. Thus $V_{l,j-1} = V_{l+1,j} \supset V_{i,j}$ and so $V_{l,j-1} \supset \mathrm{Span}(\vec{v})$, where $\vec{v}$ represents the 'variation". If $V_{l+1,j-1} \supset \mathrm{Span}(\vec{v})$, then , as l is maximal and so $V_{l+k,j-i} \neq V_{l+k+1,j}$ for all $k \geq 1$, we must have $V_{l+k,j-1} = V_{l+k-1,,j-1} \cap V_{l+k+1,j} \supset \mathrm{Span}(\vec{v})$ for $l+k \leq i-1$.
However, this implies $V_{i-1,j-1} \supset \mathrm{Span}(\vec{v}) \cup V_{i,j-1}$. As $V_{i,j}$ was the lowest dimensional set to contain $\mathrm{Span}(\vec{v})$, we must have

$$V_{i-1,j-1} = V_{i,j-1} \oplus \mathrm{Span}(\vec{v}) = V_{i,j}.$$

However, as neither $V_{i-1,j-2},\ V_{i,j-1} \subset V_{i-1,j-1}$ contain $\mathrm{Span}(\vec{v})$, we must have that

$$V_{i-1,j-2} = V_{i,j-1}$$

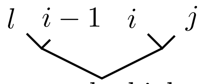
which leads to the impossible subtree:



If neither $V_{l+1,j-1}$ nor $V_{l,j-2}$ contain $\mathrm{Span}(\vec{v})$, then they must be equal, otherwise $V_{l,j-1}$ would be determined by their union. However, this also leads to an impossible subtree.

If $V_{l+1,j-1}$ does not contain $\mathrm{Span}(\vec{v})$, but $V_{l,j-2}$ does, then we can argue similarly to above that $V_{l,j-k-1} \supset \mathrm{Span}(\vec{v})$, as $V_{l+1,j-k}$ cannot. This will be true for all k until we reach a point where $V_{l,j-k-1} = V_{l+1,j-k}$. In order that this not lead to an impossibly subtree, we need $j-k \leq i-1$.

If $l = i-1$, this implies $V_{i-1,j-1} = V_{i,j}$, which, as before implies $V_{i-1,j-2} = V_{i,j-1}$ and to an impossible tree.

If $l < i-1$, then, our tree must contain 

and so $V_{l,i-2} = V_{l+1,i-1}$.

Thus $j-k \geq i-1$, as $V_{l,j-k-1} = V_{l+1,j-k}$ was the highest dimensional equality in that chain. Thus $j-k = i-1$.

Now, as $V_{i,j}$ was the lowest dimensional subspace containing $\mathrm{Span}(\vec{v})$, we then must have

$$\begin{aligned} \mathrm{dim} V_{l,i-1} &\geq \mathrm{dim} V_{i,j} \\ \Leftrightarrow l-i &\geq j-i+1 \\ \Leftrightarrow l &\geq j-1 \geq i \end{aligned}$$

an obvious contradiction. Thus, we cannot have the the root of the subtree spanning i, j is a rightmost input edge. Similarly, we can show that it cannot be a leftmost input edge and must therefore be an inner edge.

2. If $V_{i,j}$ is not in the wrap and $i = j$, we obtain a similar result.

3. If $V_{i,j}$ is in the wrap, we once again obtain an inner edge corresponding to our degree of freedom.

Thus, we get a one-to-one correspondence between inner edges and degrees of freedom, proving our result. $\square$

Unfortunately, while we obtain an identical stratification to that of [8], we do not seem to be able to define an infinitesimal composition at all. While we can easily define an operad structure on the 0-dimensional strata, as they are in one-one correspondence with binary trees, we see no way to extend this to the full space. In fact, based on correspondence with Chapoton, we do not believe it to be possible while remaining in the space of CFZ manifolds.

7 Further work and conjectures

There still fertile ground for further work in this area, which we shall now discuss. In particular there are two aspects of this discussion that I feel merit further study:

1. Realizations of generalized associahedra via subword complexes.

2. The existence of an operad containing the CFZ manifolds

7.1 Generalized associahedra

From the work of [10], we can obtain the toric variety associated to a polytope by realizing the polytope as the brick polytope of a subword complex. We in fact obtain a very neat description of this variety. But for what polytopes is this possible? In particular, for what generalized associahedra is this possible?

According to the work of [26], all generalized associahedra can be realized in this fashion.

Proposition 7.1 ([26, Prop. 1.7])**.** *Up to translation by a vector Ω, the brick polytope $B(cw_0(c), w_0)$ is obtained from the balanced W-permutahedron Perm(W) by removing all facets which do not intersect the set $\{w(q) : w \in W \text{ singleton}\}$.*

In particular, they note that the cyclohedron, the generalized associahedron of type B_n and C_n, can be realized in three dimensions at the brick polytope with $\mathbf{c} = (s_2, s_1, s_0)$. We provide a general conjecture:

Let $W_n = \langle s_0, s_1, \ldots s_{n-1} \rangle$ where s_i is the permutation matrix corresponding to the transposition $(i\ i+1)$ for $1 \leq i \leq n-1$ and s_0 is the diagonal matrix with -1 in the top left entry and $+1$ along the rest of the diagonal. The cyclohedron can then be realized as the brick polytope $B(Q, w_0)$, where our expression for w_0 is the reduced form of the expression given.

$$Q = s_0 s_1 \ldots s_{n-1} w_0 (s_0 s_1 \ldots s_{n-1})$$

$$w_0 = -\mathrm{id} = s_0 (1\ 2) s_0 (1\ 2)(1\ 3) s_0 (1\ 3) \ldots (1\ n) s_0 (1\ n).$$

We have verified this up to $n = 5$, however our approach via sorting networks will not apply and so a more general method with be required.

7.2 An operad of associahedra

We have seen that we obtain different realizations of the classical associahedron for different choices of Coxeter element $\mathbf{c}$. A neat way of classifying our choices of Coxeter element is given in [30] in terms of oriented quivers, similar to the classification of Cambrian fans by Coxeter diagrams [27]. In our case, we can uniquely describe a Coxeter element, up to reordering of commuting elements by an orientation of the graph

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n-2 \longrightarrow n-1$$

This correspondence is given by the convention

$$j \to i \Leftrightarrow s_i \text{ appears before } s_j.$$

The normal fan can be obtained from the oriented quiver as in [22] or [2], giving the polytope, while we can find the corresponding toric variety from the Coxeter element.

We can now explain why we believe the CFZ manifolds cannot be given the structure of an operad. The Loday manifolds corresponded to the Coxeter element $s_1 s_2 \dots s_{n-1}$, which gives the quiver

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n-2 \longrightarrow n-1$$

from which we can remove any segment, graft the remaining sections together and have a quiver of the same type. This implies the existence of a cooperad structure, dual to the operad structure of [8]. For the CFZ manifolds, our Coxeter element corresponds to the alternating quiver

$$1 \longrightarrow 2 \longleftarrow 3 \longrightarrow \cdots \longleftarrow n-2 \longrightarrow n-1$$

from which we cannot freely slice and graft while maintaining an alternating quiver. This leads us to conjecture that the operad of Dotsenko, Shadrin and Vallette is in fact a suboperad of an operad defined on all varieties $m_Q^{-1}(w_0 B)$ where Q is determined by $2^n - 1$ orientations of the given quiver. We also believe that attempting to discern a cooperad structure on these oriented quivers and their corresponding fans is a promising method of determining this operad structure. We end with a three last conjectures, stated somewhat informally.

Conjecture 7.2. The space of oriented quivers of type A_n forms a cooperad with the decomposition $\Delta_i(X)$ given by the formal sum of $x \otimes y$ where x is a segment of length i in X and y is the remaining segment on removal of x.

Conjecture 7.3. Removal of a segment x from X corresponding to quotienting the normal fan of X by the normal fan of x.

Conjecture 7.4. The dual of this cooperad is an operad on the space of varieties $\{m_Q^{-1}(w_0 B) : Q \text{ determined by an oriented quiver}\}$, containing the operad of [8] as a suboperad.

8 Acknowledgements

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9 Realizations of generalized associahedra

Image credit to [26], [9], [32], [4]

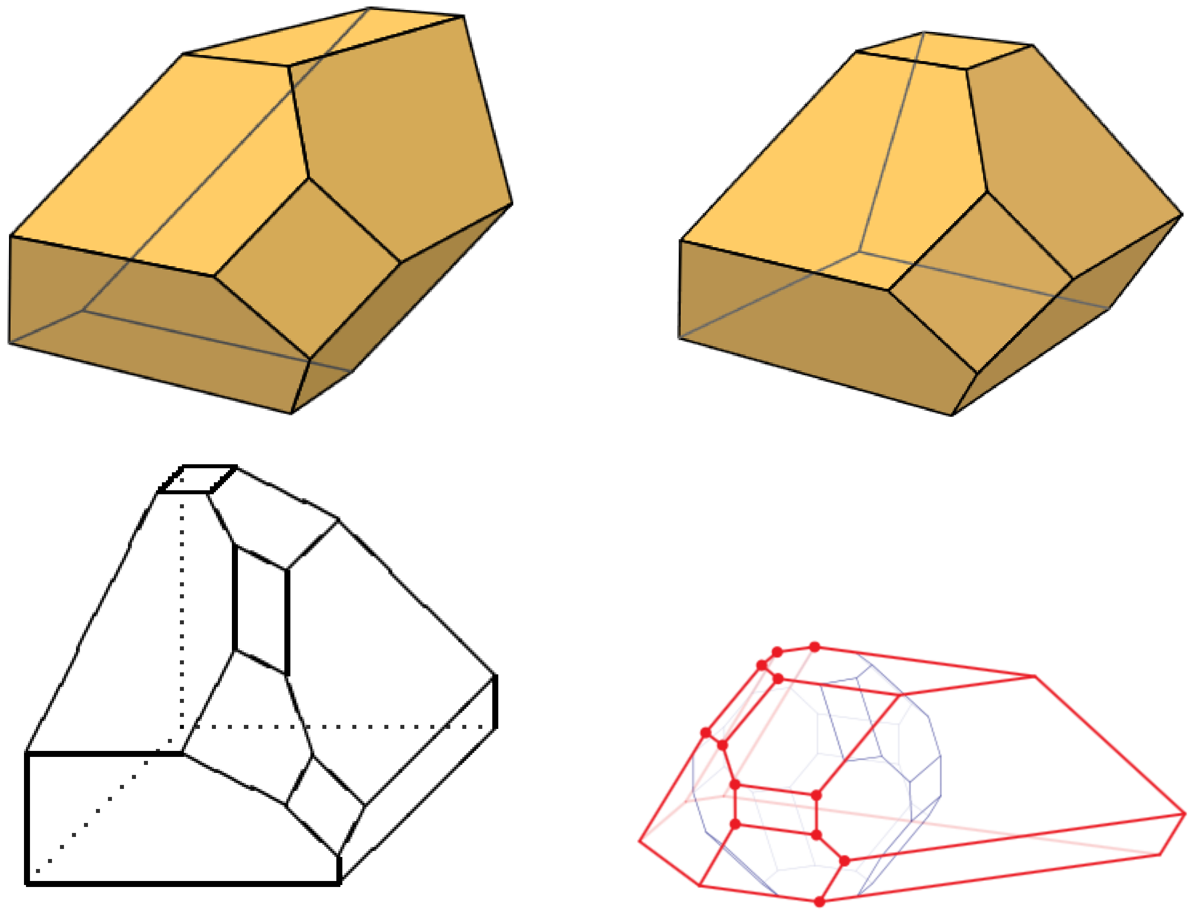

Table 1: Clockwise from the top left: The Loday associahedron, the CFZ associahedron, the cyclohedron as a brick polytope, the cyclohedron

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