Leaving Certificate Notes
Higher Level Mathematics

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Contents

1 Number Systems

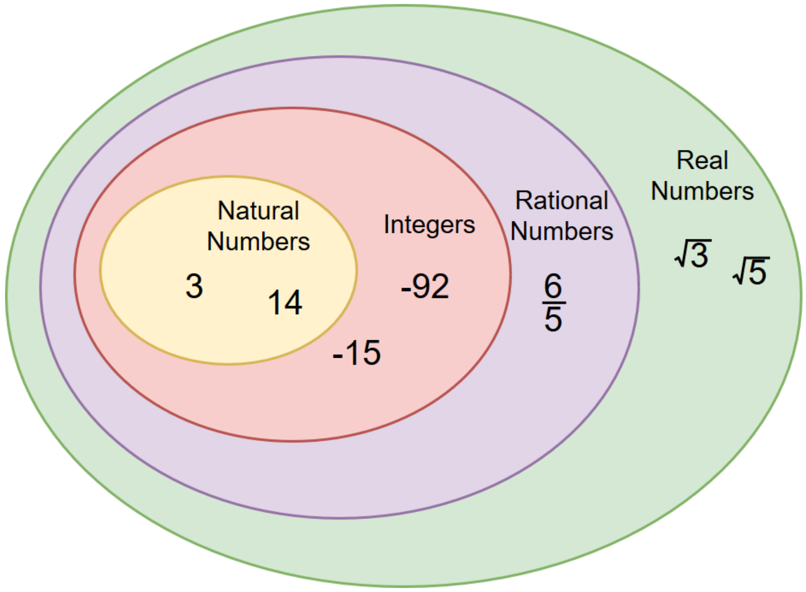

Figure 1:
A Venn diagram showing the relationships between different sets of numbers.

1.1 Number Systems

The **Natural Numbers** are the non-negative whole numbers (counting numbers). $\mathbb{N} = \{0, 1, 2, 3, ...\}$. Sometimes $\mathbb{N}^*$ is used to denote the set $\{1, 2, 3, ...\}$ without zero.

The **Integers** are all of the whole numbers, both positive and negative. $\mathbb{Z} = \{... -2, -1, 0, 1, 2, ...\}$.

The **Rational Numbers**, $\mathbb{Q}$, are the numbers generated by dividing one Integer by another (non-zero) integer. More formally, it is said that these numbers are of the form $\frac{a}{b}$, where a and b are both Integers and $b \neq 0$. All Rational Numbers can be written as decimals with either a finite number of decimal places (e.g. $\frac{1}{8} = 0.125$) or an infinite number of repeating decimal places (e.g. $\frac{1}{3} = 0.333...$). Numbers that cannot be written as a fraction are called **Irrational Numbers**, for example, $\sqrt{2}$. The Irrationals are denoted by $\mathbb{R}\backslash\mathbb{Q}$.

The set of **Real Numbers**, $\mathbb{R}$, is the set of both Rational and Irrational Numbers.

1.2 Properties of the Natural Numbers

A **Factor** of a Natural Number divides into it evenly (without leaving a remainder). For example, 2 is a factor of 6 because $6 \div 2 = 3$, which is a whole number.

The **Highest Common Factor (HCF)** of two numbers is the largest number that divides evenly into both. For example, the HCF of 12 and 18 is 6, since 6 is the largest number that divides evenly into both 12 and 18.

The **Lowest Common Multiple (LCM)** of two numbers is the smallest number that both numbers divide into evenly. For example, 36 is the LCM of 12 and 18 since $36 = 3 \times 12 = 2 \times 18$.

A **Prime Number** is a Natural Number that has only two factors. Examples of Primes are 2, 7, and 41, whose only factors are the number itself and 1.
1 is not a Prime Number because it only has one factor, and 0 is not Prime because it can be divided by any number. Zero has an infinite amount of factors. All Prime Numbers except 2 are odd, since even numbers greater than 2 can be divided by 2 by the definition of an even number.

A **Composite Number** is a whole number greater than 1 that is not Prime, such as 9, 20, and 144. These are "Composite" Numbers because they are composed, or made up of, other numbers multiplied together.

All Natural Numbers greater than 1 can be factored into Prime Numbers. For example, $9 = 3^2$ and $24 = 2^3 \times 3$. These are the **Prime Factors** of the number, and breaking down (decomposing) the number in terms of the Prime Factors is called **Prime Decomposition**.

The Fundamental Theorem of Arithmetic says that every whole number greater than 1 can be decomposed into a product of Prime Numbers in one way only. For example, $12 = 2^2 \times 3$ is the only way to break down 12 into a product of Prime Numbers.

1.3 Proof by Contradiction and a Proof that $\sqrt{2}$ is Irrational

Proof by Contradiction is a way of showing that something is true by proving that the opposite, or **converse**, leads to a logical contradiction (something that cannot be true). Since either the statement or the converse must be true and the converse yields a contradiction, then the statement must be true. This is easily seen in the proof that $\sqrt{2}$ is Irrational.

To prove that $\sqrt{2}$ is Irrational, we assume that it is not Irrational, but Rational, and show that this assumption must be false, since it leads to a contradiction.

If we assume that $\sqrt{2}$ is Rational, then that means we can write $\sqrt{2} = \frac{a}{b}$, where a and b are Integers.
We know that this fraction $\frac{a}{b}$ is in its simplest form, that is, the HCF of a and b is 1, since if they shared any other factor we could divide by it to get a smaller fraction. For example, the fraction $\frac{2}{6}$ can be reduced down since the HCF(2, 6) = 2. Dividing above and below by 2 gives $\frac{2}{6} = \frac{1}{3}$. Now, HCF(1, 3) = 1.
If we square both sides, we get $\left(\sqrt{2}\right)^2 = 2$ and $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$. Both a^2 and b^2 are Integers, since a and b are Integers.
We can also see that since $2 = \frac{a^2}{b^2}$, a^2 must be an even number. This means that a must also be even.
Since a is even, we can write it as $a = 2k$, where k is an Integer.
Then, $2 = \frac{(2k)^2}{b^2} = \frac{4k^2}{b^2}$. Multiplying by b^2, we get $2b^2 = 4k^2$, so $b^2 = 2k^2$. k^2 is an integer, so b^2, and thus b, must be even, since they are two times an integer.
We now know that both a and b are even numbers, which means they can be divided by two. This contradicts our original assumption that HCF(a, b) = 1, so our initial assumption that $\sqrt{2}$ is a Rational Number must be wrong. Thus, $\sqrt{2}$ is Irrational.

1.4 Significant Figures and Rounding

Often, answers to problems don't have to be exact. For example, we've just shown that trying to write out $\sqrt{2}$ as an exact decimal would be impossible since it's Irrational, which means it's infinitely long and doesn't repeat! Instead, an **approximation** of $\sqrt{2}$ is used. Depending on the precision of the answer needed, a different number of decimal places are used. Usually, 1.41 will do, since $1.41^2 = 1.9881$, which is close enough to 2. 1.41 is $\sqrt{2}$ rounded to 2 decimal places. If we wanted to increase the precision of the answer, we could use more decimal places. Taking $\sqrt{2}$ to three decimal places (just one more), we get 1.414. If we square this, we get $1.414^2 = 1.999396$, which is very, very close to 2!

When rounding, say, 1.414 to 1.41, you look at the third decimal place (4) and if it is less than 5, you round down, keeping the digit on the left of it. If it is 5 or greater, you round up by adding 1 to the digit on the left and then dropping that decimal place. For example, 0.125 rounded to two decimal places becomes 0.13.

When rounding to a number, we can do it to a certain number of **Significant Figures**. A Significant Figure (or Sig Fig) is one digit of the number in question. For example in the number 1729, the most significant number is the 1 because it tells us the number is in the thousands. The next most significant is 7, which tells us how many hundreds, and so on. Rounding to 2 Sig Figs in this case would mean rounding 1729 to 1700. To 3 Sig Figs would be 1730 (since the 9 is greater than 5, so we round 2 up to 3), and so on. Decimals can also be rounded in this way. 3.141 to 2 Sig Figs would be 3.1 (remember that the 3 counts as the first Sig Fig!). Zero doesn't count as a Sig Fig when it's leading the number. For example, 0.007299 rounded to 2 significant figures would be 0.0073, because 7 is the first Sig Fig, not 0.

When rounding decimals to whole numbers, often the notation "$\lceil \quad \rceil$" and "$\lfloor \quad \rfloor$" are used. These denote the "ceiling" and "floor" functions, which round up or down to the nearest whole number respectively. For example, $\lceil 3.1 \rceil = 4$, and $\lfloor 6.2 \rfloor = 6$.

1.5 Scientific Notation

There are lots of numbers in Maths, and not all of them are easy to write. For example, the speed of light in a vaccum is about 300,000,000 metres per second, and the mass of one hydrogen atom is 0.00000000000000000000000000167 kilograms! These numbers are very awkward to write out, so instead we use **Scientific Notation** to write them. This method involves using powers of 10 multiplied by some number between 1 and 10. Let's do both of these numbers as examples.

We take the number 300,000,000 and see that the only significant figure is 3. 3 is between 1 and 10 so we can leave it alone. Then, we see how many times we need to multiply 3 by to get 300,000,000. Each time we multiply by 10 we move the decimal place one place to the right. To move the decimal place from 3.0 to 300,000,000.0, we need to multiply by 10 eight times. We then write this as 3×10^8. Notice that this is the exact same number as before. All we have done is factor out 10 from 300,000,000 eight times.

For 0.00000000000000000000000000167 things are a little different. This time, we need to make the number bigger, not smaller. We see that the Sig Figs are 1, 6, and 7. To get this between 1 and 10, we have the number 1.67. We then have to divide 1.67 enough times to get the starting number. Dividing by 10 moves the decimal place one to the left. To get from 1.67 to 0.00000000000000000000000000167, we have to divide by 10 a total of 27 times. Instead of 10^{27}, which would be multiplying by 10, we write 10^{-27}, which means $\frac{1}{10^{27}}$. So, the mass of a hydrogen atom is 1.67×10^{-27} kilograms. Again, all we have done is factor out powers of 10 from the number we started with.

The power of 10 that we write for Scientific Notation is called the **Order of Magnitude** of the number. The order of magnitude of 3×10^8 is 8, and for 1.67×10^{-27} it's -27. These two numbers differ from each other by $8 + 27 = 35$ orders of magnitude.

2 Area and Volume

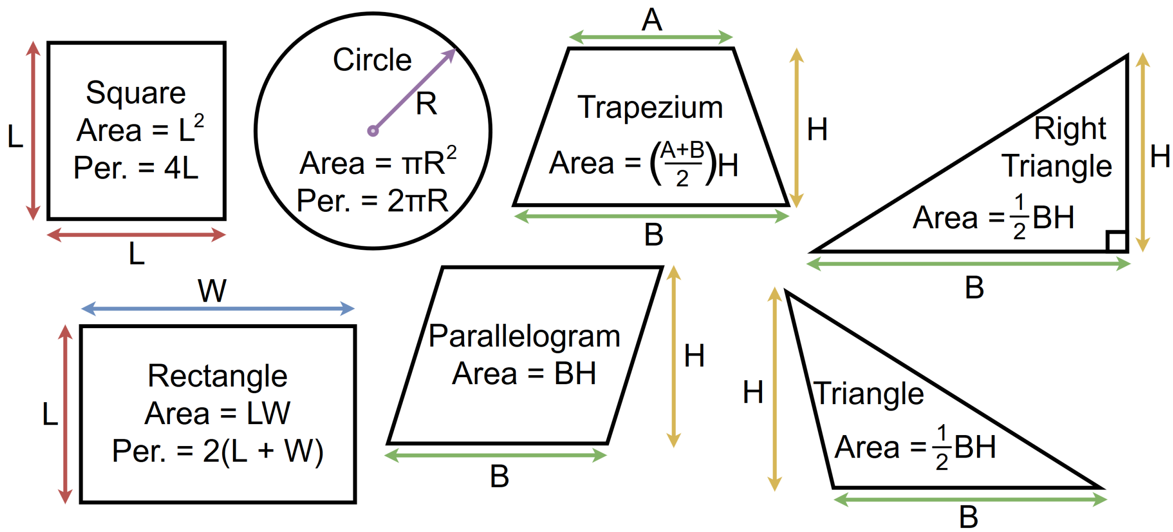

Figure 2:
Areas and perimeters of some common 2D shapes.

2.1 Area and Volume

Now that we have a good understanding of real numbers, we can start to look at some real world problems involving measurement and the sizes and shapes of different objects. We talk about shapes and objects in terms of **dimensions**. The dimension of an object is how many directions in space it takes up. For example, a square is two dimensional because you can move along the length and width of a square. A cube is three dimensional because it has length, width, and depth. **Two dimensional (2D)** shapes include triangles, squares, circles, and lots of other "flat" shapes. Each of these shapes has **three dimensional (3D)** equivalents. A square becomes a cube, a circle becomes a sphere, and a triangle becomes a pyramid, or tetrahedron. Plenty of other 3D shapes exist such as prisms, cylinders, and cones.

2.2 Regular 2D Shapes

Regular 2D shapes are made up of straight line segments. Two of the most important properties of 2D shapes are **Area** and **Perimeter**. The Area of the shape is the amount of space that the shape encloses. The Perimeter is the sum of all the side lengths of the shape. The areas and perimeters of some important shapes are shown in Figure 2. When solving problems where you must find the area of a shape, it helps to break up the area into smaller shapes that you recognise, such as rectangles and triangles. The total area will just be the areas of the smaller shapes added together.

In fact, we can use this principle of breaking up shapes into more managable pieces to prove some of the formulas we use for the areas of shapes. As an example, we'll try to prove that the area of any triangle is $A = \frac{bh}{2}$, where b is the length of the base of the triangle and h is its vertical height.

2.3 Proof of the Area of a Triangle

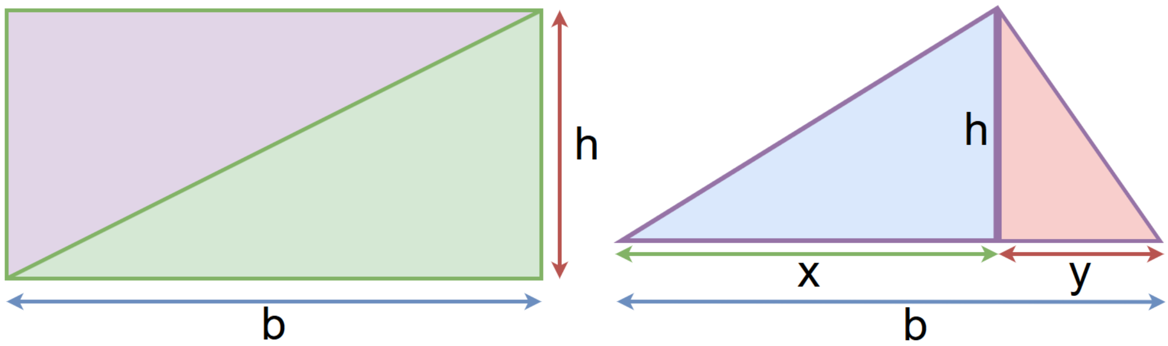

Figure 3

Let's start this proof by noticing that a right angled triangle with side lengths b and h is simply half of a rectangle with sides b and h (see Figure 3). We know the area of a rectangle is $b \times h$, so half of this area, the area of the right angled triangle, is $\frac{1}{2}bh$.

Now, if we take a scalene triangle with all sides different lengths, we don't necessarily have a right angled triangle. What we can do, however, is draw a straight line from the top of the triangle, called the apex, down to the base so that the line is at 90 degrees to the base. We've now split the scalene triangle into two right angled triangles and we know how to compute their areas.

Calling the lengths of the bases of the two right triangles x and y, we can see that the area of the scalene triangle is the sum of the areas of the two right triangles. That is, $A = A_1 + A_2$. The areas of the two right triangles are $\frac{1}{2}xh$ and $\frac{1}{2}yh$, so $A = \frac{1}{2}xh + \frac{1}{2}yh$. We can factor out $\frac{1}{2}h$ to get $A = \frac{1}{2}h(x + y)$. However, notice that the base lengths x and y add up to the total base length of the scalene triangle. That is, $b = x + y$. Since they are equal, we can replace $x + y$ with b in the equation for A to get $A = \frac{1}{2}bh$, which is the formula for the area of any triangle.

2.4 Circles

Circles are a little different to other shapes we'll look at since it's difficult to accurately measure the edge of a circle. We certainly couldn't go about it with a ruler! One of the most fundamental parts of a circle is its **Radius**. The radius is the length of the line connecting the centre of the circle to any point on the circle's edge, or **Circumference**. The **Diameter** of the circle is the length of a line passing through the centre of the circle connecting two points on the circle's circumference. The diameter is two times longer than the radius.

One of the most important, and widely used numbers in Maths, comes from the ratio of the circumference of a circle to the diameter. This number is Irrational, so instead of writing it out (which would be impossible!) we use the Greek letter pi, or π, (pronounced 'pie') in its place. This ratio of the circumference to the diameter doesn't depend on the size of the circle. π was used first about 4,000 years ago and was measured to be 3 (which is about right). That is, the circumference was three times longer than the diameter of a circle. As time went on, π was measured to better and better accuracy. Using computers, we've calculated π to trillions of decimal places! We now know that π is Irrational, and is approximately equal to 3.14159. Some common approximations for π are 3.14 and $\frac{22}{7}$. π pops up (sometimes surprisingly) in almost all areas of Maths, but what we'll focus on is how it applies directly to calculations involving circles.

We know from its definition that $\pi = \frac{\text{circumference}}{\text{diameter}}$. If we call the circumference C and the diameter d, we can multiply by d on both sides to find that the circumference (or perimeter) of the circle, $C = \pi d$. We also know that the diameter d is twice the radius, which we will call r. If $d = 2r$, then $C = 2\pi r$. We now have a formula for the circumference of any circle in terms of its radius. Deriving the formula for the area of a circle is a little more difficult and involves some calculus, so we'll leave that for a later chapter. For now, we just need that the area of a circle $A = \pi r^2$.

As well as the area of an entire circle, it's often important to be able to compute the area of part of a circle, known as a **Sector**. It's easy to see that no matter what way you rotate a circle it always looks the same. This means that we can find the area of a Sector by simply taking a fraction of a circle. For example, if we want the area of a semicircle, we just take the formula for the area of a circle, $A_{\text{circle}} = \pi r^2$ and divide by two to get $A_{\text{semicircle}} = \frac{1}{2}\pi r^2$. We could do this for any fraction we want, and indeed the formula for the area of a sector depends on how big of a slice of the circle we're talking about.

The easiest way to measure the size of a sector is to use the angle it sweeps out around the circle. We usually denote angles by the Greek letter θ, pronounced "theta". An entire circle sweeps out an angle of 360°, so it seems natural that if a sector sweeps out an angle of θ degrees, then its area will be the fraction $\frac{\theta}{360^\circ}$ of the area of the entire circle. For example, if the sector is a quarter circle then the angle $\theta = 90^\circ$ and the area is $\frac{90^\circ}{360^\circ} \times \pi r^2 = \frac{1}{4}\pi r^2$. So if we measure the angle of the sector in degrees, the area of the sector is $A = \pi r^2 \left(\frac{\theta}{360}\right)$.
It's easy to see that we can apply the exact same logic to the length of the curve, or arc, of the sector by taking a fraction of the circumference of the circle. The length l of an arc is given by $l = 2\pi r \left(\frac{\theta}{360}\right)$.

2.5 Degrees and Radians

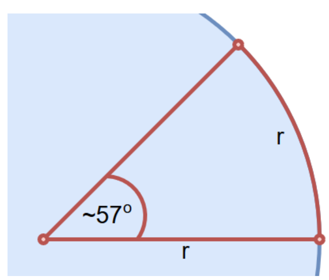

Figure 4:
The definition of one Radian

When measuring angles we're used to using **Degrees**, where we split a circle up into 360 equal parts and call each part one degree. This system works fine, but it makes some of the maths a little awkward because the measurement system isn't derived from any properties of the circle. Instead, Mathematicians prefer to use another system called **Radians**.

If we have a circle of radius r and we draw a sector so that the arc length of the sector is also r, then the angle of the sector is one Radian. One Radian is approximately 57 degrees. By defining angles in this way, an entire circle is 2π Radians.

This greatly simplifies the formulae for arc length and area of a sector. If we convert to Radians, the formula $l = 2\pi r \left(\frac{\theta}{360}\right)$ becomes $l = 2\pi r \left(\frac{\theta}{2\pi}\right) = r\theta$. The formula for the area of a sector becomes $A = \pi r^2 \left(\frac{\theta}{2\pi}\right) = \frac{1}{2}r^2\theta$.

2.6 Triangles and Pythagoras' Theorem

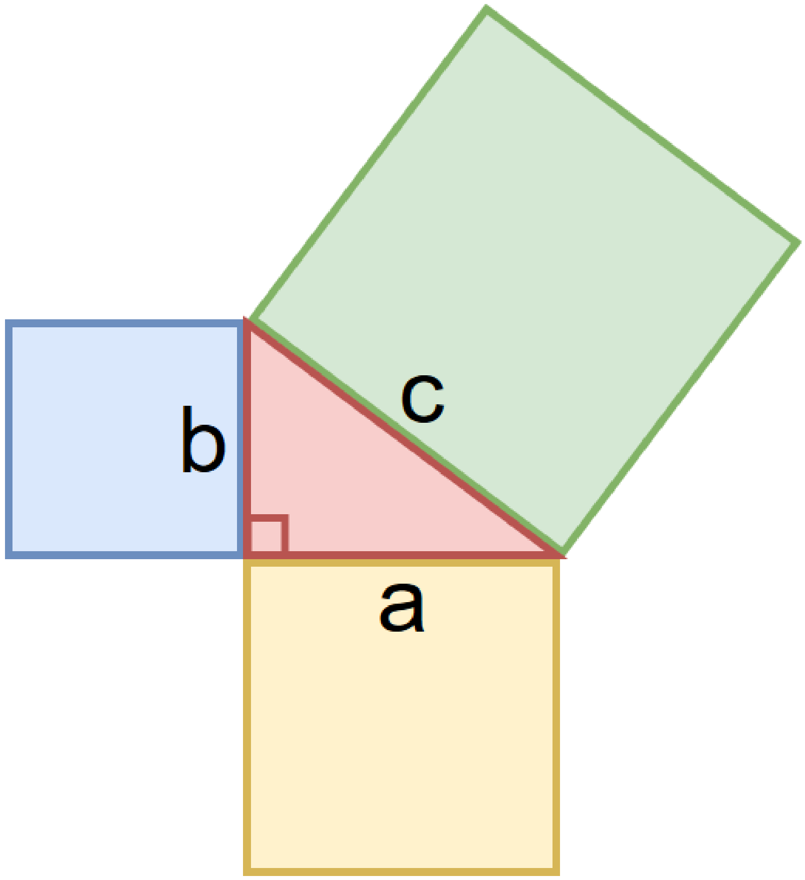

Figure 5

Going back to triangles for a bit, there's a very important formula in Maths called the **Theorem of Pythagoras**, which tells us that if you have a right angled triangle, squaring the sides at right angles to each other and adding them together equals the third side squared. The side opposite the right angle is called the **Hypotenuse**. If we call the two perpendicular sides a and b and the hypotenuse c, then Pythagoras tells us that $a^2 + b^2 = c^2$.

We can see visually that Pythagoras' Theorem means the square drawn on the side c has the same area as the sum of the area of the squares drawn on the other two sides, a and b.

2.7 Proof of the Pythagorean Theorem

There are more than 350 different proofs showing that the Pythagorean Theorem is true, but the one we'll do here is one of the most common. It uses our existing knowledge of the areas of squares and triangles.

We construct a square with side length $(a + b)$ and draw another square inside it at some angle so that the corners of the small square are on the edges of the bigger square This smaller square has side length c.

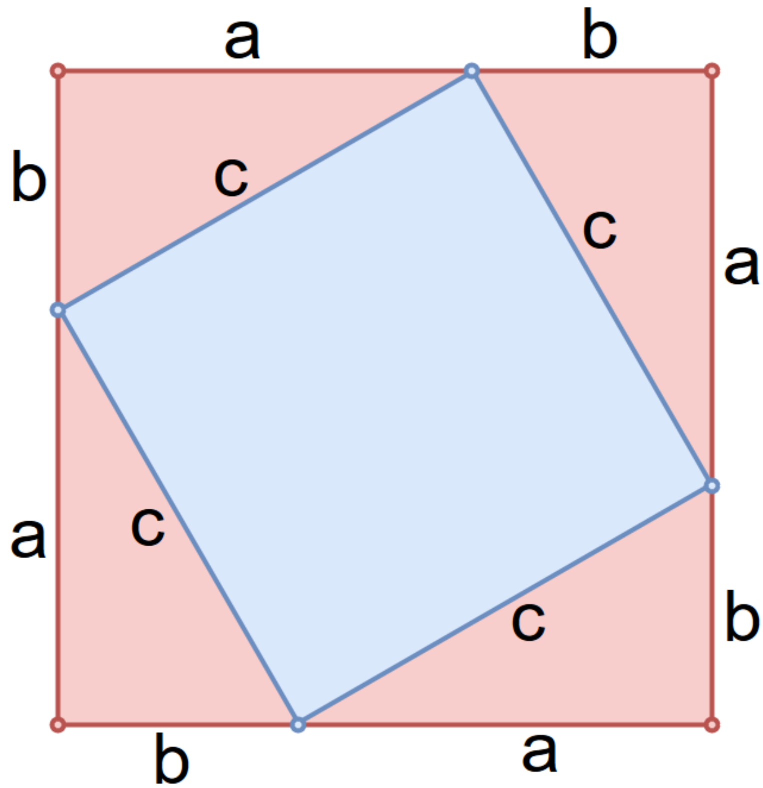

Figure 6

We now use our formula for the area of a square, and see that we can write the area of the big square as $A = l \times w = (a+b) \times (a+b) = (a+b)^2 = a^2 + b^2 + 2ab$.

However, we can also write the area of the big square as the sum of the areas of the 4 triangles and the small square. So, $A = 4 \times \frac{1}{2}ab + c^2$.

We have two different ways of writing the area A, and we know they're equal to each other. So,

$$\begin{aligned} a^2 + b^2 + 2ab &= 4 \times \frac{1}{2}ab + c^2 \\ a^2 + b^2 + 2ab &= 2ab + c^2 \\ a^2 + b^2 + 2ab - 2ab &= 2ab - 2ab + c^2 \\ a^2 + b^2 &= c^2 \end{aligned}$$

which is the statement of the Pythagorean Theorem. We've just shown that for any right angled triangle with sides a and b and hypotenuse c, the equation $a^2 + b^2 = c^2$ is true.

2.8 Regular 3D Shapes

As opposed to 2D Shapes, 3D Shapes exist in three dimensions; length, width, and height. 3D Solids are composed of three different parts; edges, vertices, and faces. **Vertices** are the points connecting **Edges**, and **Faces** are the areas enclosed by the edges. For example, a cube has six faces, twelve edges and eight vertices.

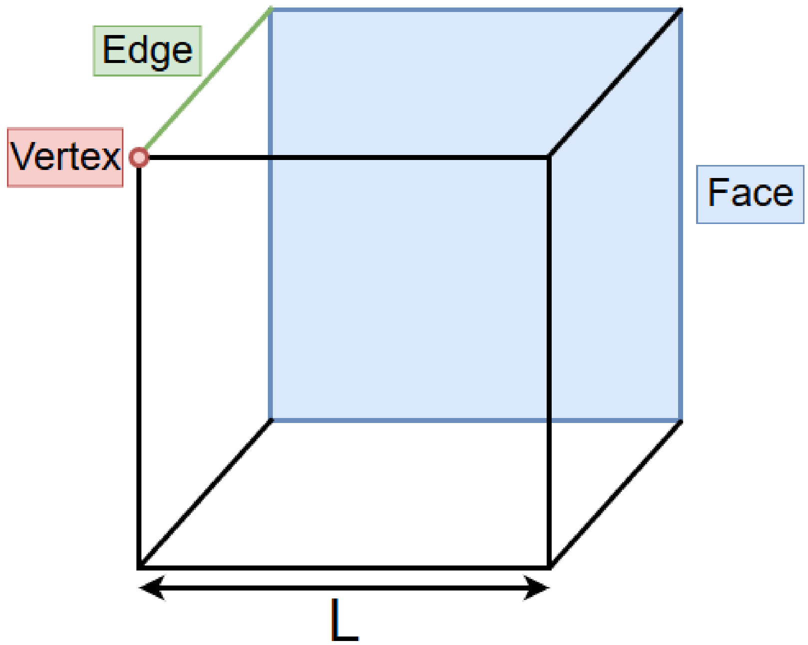

Figure 7

Volume is one of the most important properties of 3D Shapes. For a rectangular solid, the Volume is given by length $\times$ width $\times$ height, or $V = lwh$. For a cube, where all edges are the same length, the volume is l^3. There are volume formulas for most other regular shapes.

The other important property is the 3D analogue of 2D perimeter, called **Surface Area**. This is the sum of the areas of all the faces on the shape. The surface area of a cube is made up of six identical squares of side length l, so the surface area $A = 6 \times l^2 = 6l^2$.

3 Algebra I

3.1 What is Algebra?

The first thing we need to know (and to remember) is that algebra is very much something invented by people. It doesn't have some special cosmic signifigance. The system of rules we use is just a way to talk about how different things relate to each other. Even the word 'algebra' comes from the Arabic 'al-jabr' which roughly translates to 'bringinig things back together'.
The rules we have for algebra are used because they make sense and they give us correct results. That being said, algebra is by no means defined by these rules. Algebra as a subject existed for some 2500 years before it was written down in its current form. The methods for moving around little squiggles is just a convenient, yet powerful way to logically deduce answers to a whole host of problems that maths would otherwise be unequipped to deal with.

3.2 How Do I Use It?

The basic premise of Algebra is to be able to represent the different mathematical operations (addition, subtraction, multiplication, division, etc.) in a neat and convenient way and apply them to things we might not know.

For example, say we have some amount of stuff. Oranges, cars, balls of yarn, whatever. We're going to call the stuff $\square$ (yes, a square). Say we double the amount of 'square' we have. Then we have $2 \times \square$. For convenience, (and to a beginner, utter madness) we don't bother writing the multiplication sign because we're lazy. Thus, doubling the amount of stuff we have is written as $2\square$.

If we want to add stuff to other stuff, say we add another $\square$ to our $2\square$, then we just add them as we would numbers. One $\square$ plus two $\square$ is three $\square$. Otherwise known as $\square + 2\square = 3\square$. We don't bother writing the 1 in $1\square$ because, again, mathematicians are lazy.

Now, say someone comes along with some other shape, like a triangle $\triangle$, and asks us to add this triangle to our square $\square$. We can't can we? They're not the same thing! $\square + \triangle$ can't be reduced down to anything simpler. We can't add squares and triangles, or apples and oranges, or anything like that. We can only add things that are the same. This is another rule to keep in mind, and it also applies to subtraction.

Multiplication and division work as you might expect as well. $\square \times \triangle = \square\triangle$ (remember, we ignore the multiplication sign) because they're not the same thing. If we multiply two squares together however, we get that $\square \times \square = \square\square = \square^2$, or square squared (okay, maybe squares were a bad choice of shape). The little number above the $\square$ just tells us how many times we've multiplied $\square$ by itself. A paticular example could be $3^2 = 3 \times 3 = 9$.

The division sign $\div$ is almost never used in algebra. Instead, we write things as fractions. So, $\square \div \triangle = \dfrac{\square}{\triangle}$. Think of the fraction as a division sign, except the dots are replaced by numbers!

3.3 Introducing Letters

You might have guessed already that mathematicians don't usually use squares and triangles to represent unknown numbers. Instead they use letters, since words and language are mysterious concepts to mathematicians. Perfect to represent the unknown. The most commonly used letters are x and y, but the whole English alphabet is used in some shape or form. The Greek alphabet is thrown in there too, for some extra letters. Some letters are used more than others; mainly because they're nicer to write, or they stand for whatever quantity you're dealing with. For example, 'time' is almost always represented by (you guessed it) t, and 'length' by l.

3.4 Making Life Simpler

So, we now know some basic algebraic operations. What can we do with them? Well, we can start looking at ways we can use them to figure out solutions to some problems. Often, the first step in solving a problem is to try and make it look as simple as possible. That way, it might make more sense to us. Take, for example, the expression $3x + x = 16$ can be made simpler by adding the x to the $3x$. Then we get $3x + x = 4x = 16$. Now we see that if $4x = 16$, then $x = 4$ since $4 \times 4 = 16$.

4 Functions

4.1 Introduction to Functions

We learned about sets in the chapter on Real Numbers, and now we're going to to talk about how to connect sets to other sets using specific rules, called functions.

We have, in some sense, seen functions already when we looked at Arithmetic and equations to find things like the area or perimeter of different shapes. The formula $C = 2\pi r$, for the circumference of a circle, takes in some value for the radius r and gives back a value for the circumference of the circle described by that radius. This is, in essence, a function; it takes in one **input value** and gives out another **output value** which depends on the input. We can illustrate the relationship between the two using a table of values.

Radius r	Circumference C
1	$2\pi \simeq 6.28$
2	$4\pi \simeq 12.57$
4	$8\pi \simeq 25.13$
8	$16\pi \simeq 50.27$

To get the output C, we take the input r and multiply it by 2π. We could write the circumference C *as a function of* r, the radius, namely $C(r)$ (pronounced "C of r"). We can also show it graphically as a linear plot, with r on the x-axis and C on the y-axis. Of course, usually we take the radius of a circle to be positive.

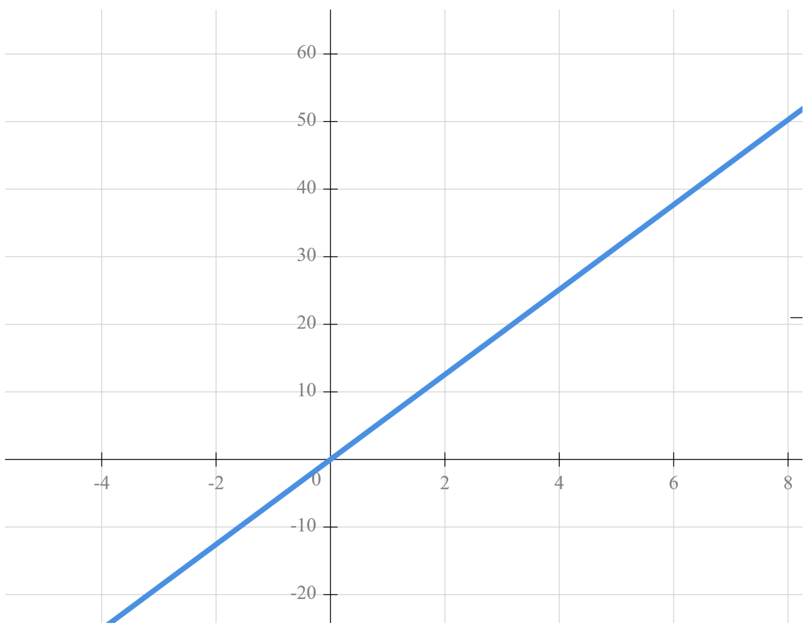

Figure 8: Graph of $y = 2\pi x$.

The set of input values, usually on the horizontal x-axis of the graph, is called the **Domain** and the set of output values, usually on the vertical y-axis of the graph, is called the **Range**, or **Codomain**.

5 Complex Numbers

5.1 Let's Break Maths

Instead of turning our number system on its head right from the start, let's try to rationalise the use of what are commonly called "imaginary" numbers, or (as they'll be referred to from now on) complex numbers.

We've already looked at solving polynomial equations, like linear equations, quadratics, and cubics. We used a few different methods, like graphing the functions, using the quadratic formula (for quadratics), and making educated guesses at solutions, especially in the case of cubics.

Let's take, as our defining example, the function

$$f(x) = x^2 + 1$$

and ask ourselves, what are the roots of this equation? That is, where is it zero? It's a second degree polynomial, so we know it must have two roots, but what are they? Let's try graphing the function $y = x^2 + 1$.

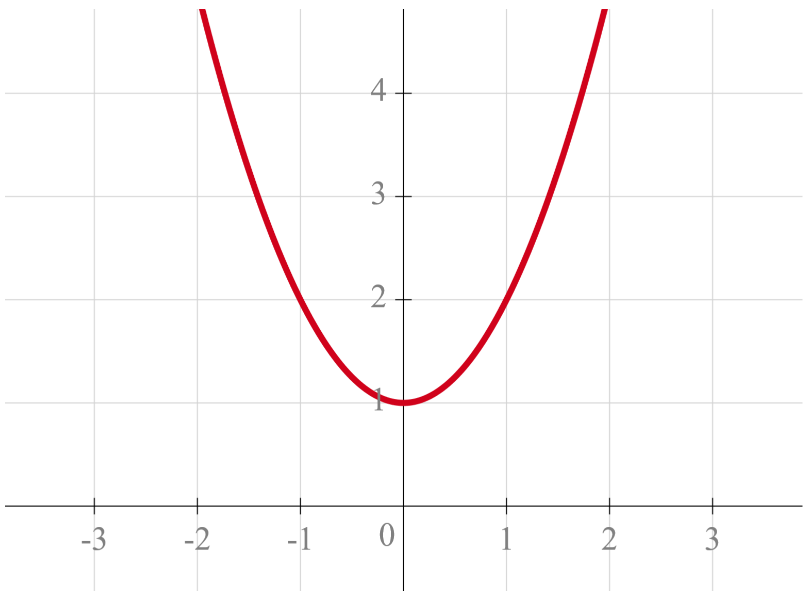

Figure 9

It seems from the graph that this equation doesn't have any roots! There's no point where the function $x^2 + 1$ crosses the horizontal x-axis. We can also try and solve the equation algebraically. We are trying to solve $x^2+1 = 0$.

$$\begin{aligned} x^2 + 1 &= 0 \\ x^2 &= -1 \\ \sqrt{x^2} &= \sqrt{-1} \\ x &= \pm\sqrt{-1} \end{aligned}$$

Oh dear, what have we gotten ourselves into? The square root of negative one? That doesn't make sense. When we think of square roots, like $\sqrt{16}$, we think "what number, when we square it, gives 16?" That number is either 4 or -4, since $4^2 = (-4)^2 = 16$. We also know that when we square a number we get a positive number, so how can we find a number that, when we square it, gives us -1?

5.2 Doing Better Than Smart People

Now, the question that we've just casually asked actually scared away some of the most intelligent minds of the 16th century. They refused to accept that equations like $x^2 + 1$ had a solution. Then again, they barely accepted negative numbers as real, so what did they know? Let's boost our own mathematical egoes and do what they dared not to do. Coming up with a sensible explanation for what's going on requires us to revisit some of what we learned when we dealt with number systems, so let's brush up on our number lines.

5.3 The Evolution of Numbers

Back in the good ol' days, things were simpler. All we had a few thousand years ago were the counting numbers; 1, 2, 3, etc. for counting cattle, days in a year, and other normal day to day things. Eventually, we needed to do more with numbers, so we started exploring things like negative numbers to deal with concepts like debt. We also eventually adopted the number zero (very controversial, I know).

Then, we started asking about dividing integers, like 1/2, or 22/7. Thus, we had developed fractions, or rational numbers. Fast forward a few hundred years and we had the irrational numbers, like $\sqrt{2}$, and eventually the transcendental numbers, which include the mathematical superstars e and π.