The Quark Propagator in Momentum Space

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The quark propagator is calculated in the Landau gauge at $\beta = 6.0$. A method for removing the dominant, tree-level lattice artefacts is presented, enabling a calculation of the momentum-dependent dynamical quark mass.

1. Introduction

The quark propagator is one of the fundamental quantities in QCD. By studying the mass function, which is the scalar part of the quark propagator, we can gain insight into the mechanisms of chiral symmetry breaking. The momentum dependence of the quark propagator is also used extensively as input in Dyson–Schwinger equations for hadronic matrix elements. A lattice study of the quark propagator may enable us to check the validity of the models used in these calulations.

2. $\mathcal{O}(a)$-improved quark propagator

All $\mathcal{O}(a)$ errors in the fermion action can be removed by adding terms to the Lagrangian [1, 2],

$$\begin{aligned}\mathcal{L}(x) = \mathcal{L}^W - \frac{i}{4}c_{sw}\,a\overline{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi \\ + \frac{b_g am}{2g_0^2}\,\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) - b_m am^2\overline{\psi}\psi \\ + c_1 a\overline{\psi}\not{D}^2\psi + c_2 am\overline{\psi}\not{D}\psi \quad (1)\end{aligned}$$

m should here be taken to be the subtracted bare mass $m \equiv m_0 - m_c$. The b_g and b_m terms correspond to a rescaling of the coupling constant and the mass respectively. The two last terms can be eliminated by a field transformation [3],

$$\begin{aligned}\psi \rightarrow \psi' &= (1+b_q am)(1-c_q a\not{D})\psi \equiv L\psi \\ \overline{\psi} \rightarrow \overline{\psi}' &= (1+b_q am)\overline{\psi}(1+c_q a\overleftarrow{\not{D}}) \equiv \overline{\psi}R \quad (2)\end{aligned}$$

The tree level improved action after the transformation (2) has $c_{sw} = 1, b_q = \frac{1}{4}$ and $c_q = \frac{1}{4}$. The improved propagator is given by

$$S(x,y) \equiv \langle\psi'(x)\overline{\psi}'(y)\rangle = L(x)S_0(x,y)R(y) \quad (3)$$

where $S_0(x,y) \equiv \langle\psi(x)\overline{\psi}(y)\rangle$. Since the propagator S_0 is defined as the inverse of the fermion matrix $M(x) \equiv \not{D}(x) + m_0 + \mathcal{O}(a)$, we can use this to obtain another, simpler expression for the improved propagator:

$$\begin{aligned}S(x,y) = \big(1+2(b_q+c_q)am\big)S_0(x,y) \\ - 2ac_q\delta(x-y) + \mathcal{O}(a^2) \quad (4)\end{aligned}$$

With this in mind, we define the tree-level improved propagator $S_I(x,y)$ as

$$S_I(x-y) \equiv (1+ma)S_0(x-y) - \frac{a}{2}\delta(x-y) \quad (5)$$

We will also introduce the tree-level 'rotated' propagator $S_R(x,y)$ as the special case of the improved propagator in (3),

$$S_R(x,y) \equiv (1+\frac{am}{2})L'(x)S_0(x,y)R'(y) \quad (6)$$

where $L' \equiv 1 - a\not{D}/4$ and $R' \equiv 1 + a\overleftarrow{\not{D}}/4$.

3. Analysis

The momentum space quark propagator in a particular gauge is given by $S(p) = \sum_x e^{-ipx}S(x,0)$. We introduce the following 'lattice momenta' $k_\mu = \frac{1}{a}\sin(p_\mu a)$ and $\hat{k}_\mu = \frac{2}{a}\sin(p_\mu a/2)$ which differ by

$$a^2\Delta k^2 \equiv \hat{k}^2 - k^2 = \frac{a^2}{4}\sum_\mu p_\mu^4 + \mathcal{O}(a^4)\,. \quad (7)$$

In the continuum, the quark propagator has the following general form,

$$S(p) = \frac{1}{i\not{p}A^c(p) + B^c(p)} \equiv \frac{Z^c(p)}{i\not{p} + M^c(p)} \quad (8)$$

We expect the lattice quark propagator to have a similar form, but with $\not{k}$ replacing $\not{p}$:

$$S(p) = \frac{Z(p)}{i\not{k} + M(p)} \tag{9}$$

The dimensionless Wilson fermion propagator at tree level is

$$S_0^{(0)}(p) = \frac{-i\not{k}a + m_0 a + \frac{1}{2}\hat{k}^2a^2}{k^2a^2 + \left(m_0 a + \frac{1}{2}\hat{k}^2a^2\right)^2}. \tag{10}$$

The tree level 'improved' propagator is given by

$$S_I^{(0)}(p) = (1 + m_0 a)S_0^{(0)}(p) - \frac{1}{2} \tag{11}$$

If we write

$$\left(S_I^{(0)}(p)\right)^{-1} = \frac{i\not{k}a + m_0 a + a\Delta M^{(0)}(p)}{Z^{(0)}(p)} \tag{12}$$

we find

$$Z^{(0)}(p) = \frac{k^2a^2(1 + m_0 a)^2 + B_{I1}^2}{(1 + m_0 a)D} \tag{13}$$

$$a\Delta M^{(0)}(p) = \frac{m_0^2a^2 - a^4\Delta k^2 + a^4\hat{k}^4/4}{1 + m_0 a} \tag{14}$$

where

$$D = k^2a^2 + \left(m_0 a + \frac{1}{2}\hat{k}^2a^2\right)^2 \tag{15}$$

$$B_{I1} = m_0 a + \frac{m_0^2a^2}{2} + \frac{a^4\Delta k^2}{2} - \frac{a^4\hat{k}^4}{8} \tag{16}$$

The corresponding tree level expressions for S_R are considerably more complicated than those for S_I, and we will not reproduce them here.

We are here primarily interested in studying the deviation of the quark propagator from its tree level value. Since QCD is asymptotically free, we should at sufficiently high momentum values get $S(p) \to S^{(0)}(p)$ up to logarithms. We will attempt to separate out the tree level behaviour by writing

$$S^{-1}(pa) = \frac{ia\not{k} + aM_r(pa) + a\Delta M^{(0)}(pa)}{Z_r(pa)Z^{(0)}(pa)} \tag{17}$$

where m_0 is replaced by m in the expressions for $Z^{(0)}$ and $\Delta M^{(0)}$. Asymptotically, we expect that $Z_r(pa) \to 1$ and $M_r(pa) \to m$ up to logarithms.

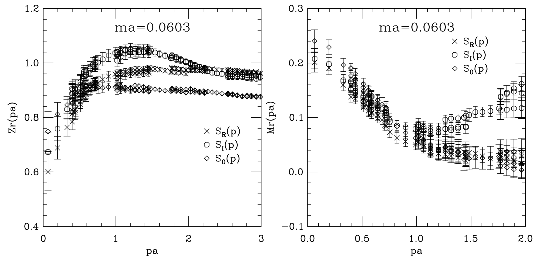

Figure 1. $Z_r(p)$ (left) and $M_r(p)$ (right) for all three propagators, from 20 configurations.

4. Results

The quark propagator is calculated at $\beta = 6.0$ on a $16^3 \times 48$ lattice, using the mean-field improved value c_{sw}=1.479. The configurations were fixed to Landau gauge with an accuracy of $\theta_{\max} = 10^{-12}$. At $\kappa = 0.137$, corresponding to $ma = 0.0603$, we have generated both S_0 and S_R. At $\kappa = 0.1381$, corresponding to $ma = 0.031$, only S_0 was generated. S_I is easily constructed from S_0.

When we calculate Z and M without factoring out the tree-level behaviour, it becomes clear that both improved propagators (S_I and S_R) are completely dominated by the unphysical tree level behaviour at high momenta. In particular, $B \equiv ZM$ computed from S_I becomes large and negative, approaching a value of $aB = -3$. Only in the infrared, below $pa \lesssim 0.8$, might we be able to extract physically significant information.

Fig. 1 shows Z_r and $M_r a$ as functions of pa for both S_0, S_I and S_R. In order to ease the comparison and remove residual anisotropy, we have selected momenta lying within one unit of spatial momentum of the 4-dimensional diagonal. While all three show a dramatic improvement on the unsubtracted data, the large momentum behaviour of the mass function from S_I is poor as a consequence of the pathological behaviour of $S_I^{(0)}(p)$ at high momenta. This is due to the cancellation of large terms in the subtraction for the mass. It is therefore desirable to use the definition S_R for

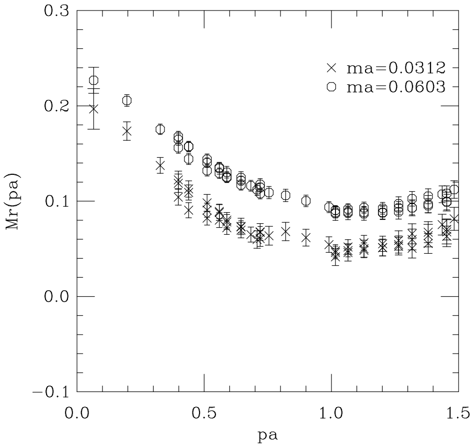

Figure 2. $M_r(p)$ from S_I, for $\kappa = 0.137$ (circles) and $\kappa = 0.381$ (crosses). The increase in $M_r(p)$ for $pa > 1$ is an indication of the difficulty of accurately subtracting off the tree level mass function. The data shown are from 50 configurations.

the improved propagator.[1]

Below $pa \sim 0.9$, the values for M_r agree within errors for the two versions of the improved propagator. In particular, the value for the infrared mass $M_r(0)$ comes out the same. In contrast, S_0 yields a mass which is 3–4σ higher.

Fig. 2 shows $M_r(p)$ calculated from S_I for both quark masses. We see that the infrared mass changes only slightly as the bare quark mass is halved, pointing to a dynamically generated quark mass of (300 ± 30)MeV in the chiral limit. Z_r turns out not to depend on the quark mass.

5. Conclusion and further work

We have used two different definitions of the $\mathcal{O}(a)$ improved quark propagator. We make use of asymptotic freedom to factor out the tree level behaviour, replacing it with the 'continuum' tree level behaviour $Z(p) = 1, M(p) = m$. This tree level subtraction dramatically improves the data. The relatively poor behaviour of the tree level subtracted S_I can be put down to the large tree level finite-a effects which require fine tuning to subtract off correctly.

For $pa \lesssim 1$, we see that $M_r(p)$ falls off with p as expected. The values obtained from S_R and from S_I are consistent, while those for the unimproved propagator S_0 differ significantly. We find that $M_r(0)$ approaches a value of 300 ± 30 MeV in the chiral limit.

We also find a significant dip in the value for $Z_r(p)$ at low momenta. This is more pronounced for the improved propagators than for the unimproved one. The next step will be a quantitative study of the functional form for Z_r and M_r

Finite volume effects have not been studied, but there is no sign of any anisotropy at low p, indicating that finite volume effects are small. Repeating these calculations at a different lattice spacing is also essential especially to get reliable results for the quark mass.

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[1] A similar conclusion was reached by C. Pittori [4].