Problem Sheet F - Solutions

(1) (i) $x\dfrac{dy}{dx} + 4y = 0$, $y(1) = 2$

$\Rightarrow \int \dfrac{dy}{y} = -4\int \dfrac{dx}{x}$

$\Rightarrow \ln y = -4\ln x + C$

$y(1) = 2 \Rightarrow \ln 2 = -4\cancel{\ln 1} + C$

$\therefore C = \ln 2$ and $\ln y = -4\ln x + \ln 2$

$\therefore \ln y - \ln 2 = -4\ln x = \ln(x^{-4})$

$\therefore \ln\left(\dfrac{y}{2}\right) = \ln\left(\dfrac{1}{x^4}\right)$, so

$\dfrac{y}{2} = \dfrac{1}{x^4}$ or $y = \dfrac{2}{x^4}$.

(ii) $\dfrac{dy}{dx} - (\tan x)\, y = 0$, $y(0) = 1$

$\Rightarrow \int \dfrac{dy}{y} = \int \tan x\, dx$

$\Rightarrow \ln y = -\ln(\cos x) + C$

$y(0) = 1 \Rightarrow \ln 1 = -\ln(\cos 0) + C$

i.e. $\cancel{\ln 1} = -\cancel{\ln 1} + C$ $\therefore C = 0$

$\therefore \ln y = -\ln(\cos x) = \ln\left(\dfrac{1}{\cos x}\right)$

and so $y = \dfrac{1}{\cos x}$.

2 (i) $\frac{dy}{dt} = ky$, $y(0) = 1,000$.

$(k = \frac{5}{100} = \frac{1}{20})$

(ii) $\int \frac{dy}{y} = k \int dt$

$\Rightarrow \ln y = kt + c$

$\Rightarrow y(t) = e^{kt} \cdot e^{c} = D e^{kt}$

$y(0) = 1,000 \Rightarrow 1,000 = D$, so

$y(t) = 1,000\, e^{kt}$.

(iii) $2,000 = 1,000\, e^{kT} \Rightarrow 2 = e^{kT}$, so

$kT = \ln 2$ OR $T = \frac{\ln 2}{k} = \frac{\ln 2}{\frac{1}{20}} = 20 \ln 2$ mins

(iv) $20,000 = 1,000\, e^{kt}$

$\Rightarrow 20 = e^{kt}$ OR $kt = \ln 20$, so

$t = \frac{\ln 20}{k} = 20 \ln 20$ mins.

3 $y(t) = D e^{kt}$, $D = y(0)$; call this y_0.

$\therefore 5 \cancel{y_0} = \cancel{y_0} e^{k \cdot 20}$

$\Rightarrow \ln 5 = k \cdot 20$, so $k = \frac{\ln 5}{20}$.

Then $2 \cancel{y_0} = \cancel{y_0} e^{kT}$ OR

$kT = \ln 2$

$\Rightarrow T = \frac{\ln 2}{k} = \frac{\ln 2}{\frac{\ln 5}{20}} = 20 \frac{\ln 2}{\ln 5}$ mins.

Problem Sheet 6 – Solutions

(i)
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right) \xrightarrow[R_3-8R_1]{R_2+2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array}\right) \xrightarrow{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{7}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & 3/7 & -\frac{1}{7} \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\therefore \quad \begin{aligned} x_1 \quad + \tfrac{3}{7}x_3 &= -\tfrac{1}{7} \\ x_2 + 4/7\, x_3 &= 1/7 \end{aligned}$$

Letting $x_3 = t$ gives solns. of form
$$x_1 = -\tfrac{1}{7} - \tfrac{3}{7}t, \quad x_2 = \tfrac{1}{7} - \tfrac{4}{7}t, \quad x_3 = t.$$

Infinitely many solns.

(ii)
$$\left(\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array}\right) \xrightarrow{R_{2\leftrightarrow 3}} \left(\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 3 & 1 & 3 & 11 \\ 5 & 3 & 2 & 0 \\ -6 & -4 & 2 & 30 \end{array}\right)$$

$$\xrightarrow[R_4+2R_1]{R_2-R_1} \left(\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 4 & 26 \\ 5 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{3}R_1} \left(\begin{array}{ccc|c} 1 & \frac{2}{3} & -\frac{1}{3} & -5 \\ 0 & 1 & -4 & -26 \\ 5 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_3-5R_1]{} \left(\begin{array}{ccc|c} 1 & 2/3 & -1/3 & -5 \\ 0 & 1 & -4 & -26 \\ 0 & -\frac{1}{3} & 11/3 & 25 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-3R_3} \left(\begin{array}{ccc|c} 1 & \frac{2}{3} & -\frac{1}{3} & -5 \\ 0 & 1 & -4 & -26 \\ 0 & 1 & -11 & -75 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_3-R_2]{} \left(\begin{array}{ccc|c} 1 & \frac{2}{3} & -\frac{1}{3} & -5 \\ 0 & 1 & -4 & -26 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[-\frac{1}{7}R_3]{} \left(\begin{array}{ccc|c} 1 & \frac{2}{3} & -\frac{1}{3} & -5 \\ 0 & 1 & -4 & -26 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1-\frac{2}{3}R_2} \left(\begin{array}{ccc|c} 1 & 0 & 7/3 & \frac{37}{3} \\ 0 & 1 & -4 & -26 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2-4R_3} \left(\begin{array}{ccc|c} 1 & 0 & 7/3 & 37/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 - \frac{7}{3}R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \therefore \; x_1 = -4, \; x_2 = 2, \; x_3 = 7$$

Unique soln

(2)

(iii)

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & 5 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - R_1} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & -10 & -12 & -12 & 4 \end{array}\right)$$

$$\xrightarrow{R_3 + 2R_2} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right)$$

No soln.