Problem Sheet E – Solutions ①

① (i) $y = e^{x\tan x}$

$$\frac{dy}{dx} = e^{x\tan x}.(1.\tan x + x.\sec^2 x)$$

(ii) $y = \cos(\ln x)$

$$\frac{dy}{dx} = -\sin(\ln x).\frac{1}{x}$$

(iii) $y = x^3.\log_2(3-2x) = x^3.\dfrac{\ln(3-2x)}{\ln 2}$

$$\frac{dy}{dx} = \frac{1}{\ln 2}\left(x^3.\frac{1}{3-2x}.(-2) + 3x^2.\ln(3-2x)\right)$$

(iv) $y = e^{\sqrt{1+5x^3}}$

$$\frac{dy}{dx} = e^{\sqrt{1+5x^3}}.\left(\tfrac{1}{2}(1+5x^3)^{-\frac{1}{2}}.15x^2\right)$$

(v) $y = \ln(\cos(e^x))$

$$\frac{dy}{dx} = \frac{1}{\cos(e^x)}.-\sin(e^x).e^x.$$

② (i) $y = e^{\frac{1}{x}} \Rightarrow \dfrac{dy}{dx} = e^{\frac{1}{x}}.\left(\dfrac{-1}{x^2}\right) = \dfrac{-e^{\frac{1}{x}}}{x^2}.$

(ii) $y = 3^{-x} \Rightarrow \ln y = -x\ln 3$

$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = -\ln 3 \Rightarrow \frac{dy}{dx} = -\ln 3.y = -\ln 3.3^{-x}.$

(iii) $y = \pi^{\sin x} \Rightarrow \ln y = \sin x.\ln\pi$

$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \cos x.\ln\pi \Rightarrow \frac{dy}{dx} = \ln\pi.\cos x.y$

$= \ln\pi.\cos x.\pi^{\sin x}.$

(iv) $y = x^{\sin x} \Rightarrow \ln y = \sin x \cdot \ln x$ ②

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = y\left(\cos x \ln x + \frac{\sin x}{x}\right) = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$.

(v) $e^{xy} + x = 1 \Rightarrow e^{xy}\left(1\cdot y + x\cdot \frac{dy}{dx}\right) + 1 = 0$

$\Rightarrow y + x\frac{dy}{dx} = -e^{-xy}$

$\Rightarrow \frac{dy}{dx} = -\frac{(e^{-xy} + y)}{x}$.

③ (i) $I = \int_0^{\frac{\pi}{4}} e^{\tan x} \cdot \sec^2 x \, dx$;

Let $u = \tan x$; $du = \sec^2 x \, dx$

$x = 0 \Rightarrow u = 0$, $x = \frac{\pi}{4} \Rightarrow u = 1$

$\therefore I = \int_0^1 e^u \, du = (e^u)\Big|_0^1 = e - 1$.

(ii) $I = \int_0^1 e^{2x}\sqrt{1+e^{2x}}\, dx$;

Let $u = 1 + e^{2x}$; $du = 2e^{2x}dx$

$x = 0 \Rightarrow u = 2$, $x = 1 \Rightarrow u = 1 + e^2$

$\therefore I = \frac{1}{2}\int_2^{1+e^2} u^{1/2} \, du = \frac{1}{2}\cdot\frac{2}{3}\left(u^{3/2}\right)\Big|_2^{1+e^2}$

$= \frac{1}{3}\left((1+e^2)^{3/2} - (2)^{3/2}\right)$.

(iii) $I = \int_0^1 \frac{e^x \, dx}{e^x + 10}$;

Let $u = e^x + 10$; $du = e^x dx$

$x = 0 \Rightarrow u = 11$, $x = 1 \Rightarrow u = e + 10$

$\therefore I = \int_{11}^{e+1} \frac{du}{u} = (\ln u)\Big|_{11}^{e+1} = \ln(e+1) - \ln(11)$

$= \ln\left(\frac{e+1}{11}\right)$.