Problem Sheet C - Solutions

(1) (i) $y = \sin^2\left(\frac{x^2}{x^2+1}\right)$

$$\frac{dy}{dx} = 2\sin\left(\frac{x^2}{x^2+1}\right).\cos\left(\frac{x^2}{x^2+1}\right).\left(\frac{(x^2+1).2x - x^2.(2x)}{(x^2+1)^2}\right)$$

(ii) $y = x^3(\cos x + \sin 3x)^{\frac{1}{3}}$

$$\frac{dy}{dx} = 3x^2(\cos x + \sin 3x)^{\frac{1}{3}} + \frac{1}{3}x^3(\cos x + \sin 3x)^{-\frac{2}{3}}.(-\sin x + 3\cos 3x)$$

(iii) $y = \frac{x\tan x}{x+\tan x}$

$$\frac{dy}{dx} = \frac{(x+\tan x)(1.\tan x + x\sec^2 x) - x\tan x(1+\sec^2 x)}{(x+\tan x)^2}.$$

(2) (i) $I = \int \frac{x^2 - 2x^3}{x^5}\,dx = \int (x^{-3} - 2x^{-2})\,dx$

$$= \frac{x^{-2}}{-2} + 2x^{-1} + C = \frac{2}{x} - \frac{1}{2x^2} + C.$$

(ii) $I = \int \frac{3x^2+6x-5}{(x^3+3x^2-5x+1)^5}\,dx$

Let $u = x^3+3x^2-5x+1$

$\therefore \frac{du}{dx} = 3x^2+6x-5$ or $du = (3x^2+6x-5)\,dx$,

$$\therefore I = \int \frac{du}{u^5} = \int u^{-5}\,du = \frac{u^{-4}}{-4} + C = -\frac{1}{4u^4} + C$$

$$= \frac{-1}{4(x^3+3x^2-5x+1)^4} + C.$$

(iii) $I = \int x(2-x^2)^3\,dx$

Let $u = 2-x^2$, $\therefore \frac{du}{dx} = -2x$ or $du = -2x\,dx$,

$$\therefore I = -\frac{1}{2}\int u^3\,du = -\frac{1}{8}u^4 + C = \frac{-(2-x^2)^4}{8} + C.$$

(iv) $I = \int \frac{x\,dx}{\sqrt{4-5x^2}}$

Let $u = 4-5x^2$, $\therefore \frac{du}{dx} = -10x$ or $du = -10x\,dx$,

$$\therefore I = -\frac{1}{10}\int u^{-\frac{1}{2}}\,du = -\frac{1}{5}u^{\frac{1}{2}} + C$$

$$= -\frac{\sqrt{4-5x^2}}{5} + C.$$

(3) (i) $I = \int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\,dx$

Let $u = \sqrt{x}$; $\therefore \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ or

$du = \frac{dx}{2\sqrt{x}}$;

$\therefore I = 2\int \sec^2 u\,du = 2\tan u + C = 2\tan\sqrt{x} + C.$

(ii) $I = \int x \sec^2(x^2)\,dx$

Let $u = x^2$; $\therefore \frac{du}{dx} = 2x$ or $du = 2x\,dx$;

$\therefore I = \frac{1}{2}\int \sec^2 u\,du = \frac{1}{2}\tan u + C = \frac{1}{2}\tan(x^2) + C.$

iii) $I = \int_0^{\pi/4} \sec^2 x \tan x\,dx$

Let $u = \tan x$; $\therefore \frac{du}{dx} = \sec^2 x$ or

$du = \sec^2 x\,dx$;

$x = 0 \Rightarrow u = \tan 0 = 0$

$x = \pi/4 \Rightarrow u = \tan \pi/4 = 1$

$\therefore I = \int_0^1 u\,du = \left(\frac{1}{2}u^2\right)\Big|_0^1 = \frac{1}{2}.$

iv) $I = \int_0^2 x \cos(3x^2)\,dx$

Let $u = 3x^2$; $\therefore \frac{du}{dx} = 6x$ or $du = 6x\,dx$

$x = 0 \Rightarrow u = 0$

$x = 2 \Rightarrow u = 3.(2)^2 = 12$

$\therefore I = \frac{1}{6}\int_0^{12} \cos u\,du = \frac{1}{6}(\sin u)\Big|_0^{12} = \frac{1}{6}(\sin 12 - \sin 0)$

$= \frac{1}{6}\sin 12.$

Problem Sheet D — Solutions

① (i) $I = \int_0^{\pi/4} \tan^3 x \sec^2 x\,dx$;

Let $u = \tan x$; $\frac{du}{dx} = \sec^2 x$ OR $du = \sec^2 x\,dx$.

$x = 0 \Rightarrow u = 0$, $x = \frac{\pi}{4} \Rightarrow u = 1$.

$\therefore I = \int_0^1 u^3\,du = \frac{1}{4}u^4 \Big|_0^1 = \frac{1}{4}$.

(ii) $I = \int_0^{\pi/2} \cos^3 x \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x \sin^2 x \cos x\,dx$

$= \int_0^{\pi/2} (1 - \sin^2 x) \sin^2 x \cos x\,dx$;

Let $u = \sin x$; $\frac{du}{dx} = \cos x$ or $du = \cos x\,dx$.

$x = 0 \Rightarrow u = 0$, $x = \frac{\pi}{2} \Rightarrow u = 1$.

$\therefore I = \int_0^1 (1-u^2)\,u^2\,du = \int_0^1 (u^2 - u^4)\,du$

$= \left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right)\Big|_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$.

(iii) $I = \int_1^2 x\sqrt{x-1}\,dx$;

Let $u = x - 1$; $\frac{du}{dx} = 1$ or $du = dx$.

Also $x = u + 1$; $x = 1 \Rightarrow u = 0$, $x = 2 \Rightarrow u = 1$.

$\therefore I = \int_0^1 (u+1)\sqrt{u}\,du = \int_0^1 (u^{3/2} + u^{1/2})\,du$

$= \left(\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right)\Big|_0^1 = \frac{2}{5} + \frac{2}{3} = 1\frac{1}{15}$.

(iv) $I = \int_0^{\pi/3} \frac{\sin x}{\cos^2 x}\,dx$;

Let $u = \cos x$; $\frac{du}{dx} = -\sin x$ or $du = -\sin x\,dx$.

$x = 0 \Rightarrow u = 1$, $x = \pi/3 \Rightarrow u = \frac{1}{2}$.

$\therefore I = -\int_1^{1/2} \frac{du}{u^2} = \int_{1/2}^1 u^{-2}\,du = \left(\frac{u^{-1}}{-1}\right)\Big|_{1/2}^1$

$= \left(-\frac{1}{u}\right)\Big|_{1/2}^1 = \left(-\frac{1}{1}\right) - \left(-\frac{1}{1/2}\right) = -1 + 2 = 1$.

2 (i) $I = \int_0^8 x\sqrt{1+x}\,dx$

Let $u = 1+x$; $\therefore \frac{du}{dx} = 1$ or $du = dx$

Also $x = u-1$;

$x = 0 \Rightarrow u = 1$

$x = 8 \Rightarrow u = 9$

$\therefore I = \int_1^9 (u-1)u^{1/2}\,du = \int_1^9 (u^{\frac{3}{2}} - u^{\frac{1}{2}})\,du$

$= \left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right)\Big|_1^9 = \left\{\left(\frac{2}{5}(9)^{\frac{5}{2}} - \frac{2}{3}(9)^{\frac{3}{2}}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right\}$

$= \left\{\left(\frac{2}{5}\cdot 3\cdot(9)^2 - \frac{2}{3}\cdot 3(9)\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right\} = ?$

(ii) $I = \int_1^2 \frac{x+2}{\sqrt{x^2+4x+7}}\,dx$

Let $u = x^2+4x+7$; $\therefore \frac{du}{dx} = 2x+4 = 2(x+2)$

or $du = 2(x+2)\,dx$.

$x = 1 \Rightarrow u = 1^2 + 4(1) + 7 = 12$

$x = 2 \Rightarrow u = 2^2 + 4(2) + 7 = 19$;

$\therefore I = \frac{1}{2}\int_{12}^{19} \frac{du}{\sqrt{u}} = \frac{1}{2}\int_{12}^{19} u^{-\frac{1}{2}}\,du = (\sqrt{u})\Big|_{12}^{19} = \sqrt{19} - \sqrt{12}$.

(iii) $I = \int_{\pi^2}^{4\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Let $u = \sqrt{x}$; $\therefore \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

or $du = \frac{1}{2\sqrt{x}}dx$.

$x = \pi^2 \Rightarrow u = \pi$

$x = 4\pi^2 \Rightarrow u = 2\pi$

$\therefore I = 2\int_{\pi}^{2\pi} \sin u\,du = 2(-\cos u)\Big|_{\pi}^{2\pi}$

$= 2\{-\cos 2\pi + \cos\pi\} = 2\{-1-1\} = -4$.

3) (i) $I = \int_0^e \frac{dx}{x+e}$

Let $u = x+e$; $\therefore \frac{du}{dx} = 1$ or $du = dx$

$x = 0 \Rightarrow u = e$

$x = e \Rightarrow u = e+e = 2e$

$\therefore I = \int_e^{2e} \frac{du}{u} = (\ln u)\Big|_e^{2e} = \ln 2e - \ln e$

$= \ln\left(\frac{2e}{e}\right) = \ln 2.$

ii) $I = \int_1^2 \frac{5x^4}{x^5+1}\, dx$

Let $u = x^5+1$; $\therefore \frac{du}{dx} = 5x^4$ or $du = 5x^4\, dx$

$x = 1 \Rightarrow u = 2$

$x = 2 \Rightarrow u = 2^5+1 = 33$

$\therefore I = \int_2^{33} \frac{du}{u} = (\ln u)\Big|_2^{33} = \ln 33 - \ln 2 = \ln\left(\frac{33}{2}\right)$

iii) $I = \int_2^4 \frac{(\ln x)^2}{x}\, dx$

Let $u = \ln x$; $\therefore \frac{du}{dx} = \frac{1}{x}$ or $du = \frac{1}{x}\, dx$

$x = 2 \Rightarrow u = \ln 2$

$x = 4 \Rightarrow u = \ln 4$

$\therefore I = \int_{\ln 2}^{\ln 4} u^2\, du = \left(\frac{1}{3} u^3\right)\Big|_{\ln 2}^{\ln 4}$

$= \frac{1}{3}\left((\ln 4)^3 - (\ln 2)^3\right).$