Problem Sheet A - Solutions.

1. (i) $\left(\frac{8}{125}\right)^{-\frac{1}{3}} = \left(\frac{125}{8}\right)^{1/3} = \left(\left(\frac{5}{2}\right)^3\right)^{\frac{1}{3}} = \frac{5}{2}$

(ii) $64^{\frac{5}{6}} = (2^6)^{\frac{5}{6}} = 2^5 = 32$

(iii) $3\log_a 2 + \log_a 8 - \log_a 4 = 3\log_a 2 + \log_a(2^3) - \log_a(2^2)$

$= 3\log_a 2 + 3\log_a 2 - 2\log_a 2 = 4\log_a 2.$

(iv) $\frac{1}{2}\ln 7 + 2\ln 3 = \ln\sqrt{7} + \ln 9 = \ln(9\sqrt{7}).$

2. (i) $\log_a 5 + \log_a(x-3) = \log_a(2x-3)$

$\Rightarrow \log_a 5 = \log_a(2x-3) - \log_a(x-3) = \log_a\left(\frac{2x-3}{x-3}\right)$

$\therefore \frac{2x-3}{x-3} = 5$, so $2x-3 = 5x-15$

and so $3x = 12$ or $x = 4$.

(ii) $2\log_{10}x - \log_{10}(20-x) = 1$

$\Rightarrow \log_{10}(x^2) - \log_{10}(20-x) = 1$

$\Rightarrow \log_{10}\left(\frac{x^2}{20-x}\right) = 1 \Rightarrow \frac{x^2}{20-x} = 10^1 = 10$

$\Rightarrow x^2 = 200 - 10x \Rightarrow x^2 + 10x - 200 = 0$

$\Rightarrow (x+20)(x-10) = 0 \quad \therefore x = 10$

($\log_{10}(-20)$ is not defined, so $x \neq -20$).

(iii) $\ln x = 2 + \ln 3 \Rightarrow \ln x - \ln 3 = 2$

$\Rightarrow \ln\left(\frac{x}{3}\right) = 2 \Rightarrow \frac{x}{3} = e^2 \Rightarrow x = 3e^2.$

(iv) $\log_2(x^{\frac{5}{2}}) + \log_2(x^{\frac{1}{2}}) = 6$

$\Rightarrow \log_2\left(x^{\frac{5}{2}} \cdot x^{\frac{1}{2}}\right) = 6$

$\Rightarrow \log_2(x^3) = 6 \Rightarrow x^3 = 2^6 = 64$

$\Rightarrow x = 4.$

3. $f(x) = \sqrt{x-2}$, $g(x) = x^2 + 3$.

$(f \circ g)(x) = f(g(x)) = f(x^2+3) = \sqrt{x^2+3-2}$
$= \sqrt{x^2+1}$.

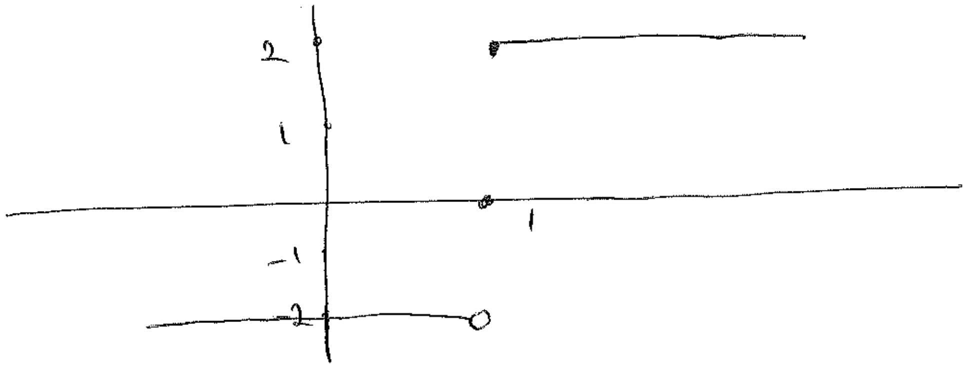
$(g \circ f)(x) = g(f(x)) = g(\sqrt{x-2}) = (x-2)+3 = x+1$.

4. (i) $\lim_{x \to 2} (x^3 + 3x^2 - x - 5) = 2^3 + 3.2^2 - 2 - 5 = 13$.

(ii) $\lim_{x \to 3} \frac{x^2 - 3x + 2}{2x^2 + x - 5} = \frac{3^2 - 3.3 + 2}{2.3^2 + 3 - 5} = \frac{2}{16} = \frac{1}{8}$.

(iii) $\lim_{x \to 2} \frac{5x^3 + 4}{x - 3} = \frac{5.2^3 + 4}{2 - 3} = -44$.

5.



No and no!

$\lim_{x \to 1^+} f(x) = 2$ but $\lim_{x \to 1^-} f(x) = -2$.

By def. $f(x)$ is not cont. at $x = 1$.

Problem Sheet B - Solutions

1 (i) $$\lim_{x\to 2}\frac{2-x}{4-x^2} = \lim_{x\to 2}\frac{\cancel{2-x}}{\cancel{(2-x)}(2+x)} = \frac{1}{4}.$$

(ii) $$\lim_{x\to -3}\frac{x^2-x-12}{x^2+2x-3} = \lim_{x\to -3}\frac{\cancel{(x+3)}(x-4)}{\cancel{(x+3)}(x-1)} = \frac{7}{4}.$$

(iii) $$\lim_{x\to \infty}\frac{\sqrt{36x^2-100}}{5-x} = \lim_{x\to\infty}\frac{\sqrt{36-\frac{100}{x^2}}}{\frac{5}{x}-1}$$

$$= \frac{\sqrt{36}}{-1} = -6.$$

2 (i) $$y = \frac{\sqrt{x}-1}{\sqrt{x}+1} \Rightarrow \frac{dy}{dx} = \frac{(\sqrt{x}+1)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)-(\sqrt{x}-1)\frac{1}{2}x^{-\frac{1}{2}}}{(\sqrt{x}+1)^2}$$

$$= \frac{1+\frac{1}{\sqrt{x}}-1+\frac{1}{\sqrt{x}}}{2(\sqrt{x}+1)^2} = \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}.$$

(ii) $$y = \left(\frac{x^2}{8}+x-\frac{1}{x}\right)^4$$

$$\Rightarrow \frac{dy}{dx} = 4\left(\frac{x^2}{8}+x-\frac{1}{x}\right)^3.\left(\frac{x}{4}+1+\frac{1}{x^2}\right).$$

(iii) $$y = (4x+3)^4(x+1)^{-3}$$

$$\Rightarrow \frac{dy}{dx} = 4(4x+3)^3.4.(x+1)^{-3}+(4x+3)^4.(-3(x+1)^{-4}).$$

3 (i) $$x^2y+3xy^3-x = 3$$

$$\Rightarrow 2x.y+x^2.\frac{dy}{dx}+3.y^3+9x.y^2.\frac{dy}{dx}-1=0$$

$$\Rightarrow \frac{dy}{dx}(x^2+9xy^2) = 1-2xy-3y^3$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-2xy-3y^3}{x^2+9xy^2}.$$

(ii) $x^3y^2 - 5x^2y + x = 1$ ②

$$\Rightarrow 3x^2 \cdot y^2 + 2x^3 \cdot y \cdot \frac{dy}{dx} - 10x \cdot y - 5x^2 \cdot \frac{dy}{dx} + 1 = 0$$

$$\Rightarrow \frac{dy}{dx}\left(2x^3y - 5x^2\right) = -1 + 10xy - 3x^2y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1 + 10xy - 3x^2y^2}{2x^3y - 5x^2}.$$

(iii) $\frac{1}{x} + \frac{1}{y} = 4 \Rightarrow y + x = 4xy$

$$\Rightarrow \frac{dy}{dx} + 1 = 4y + 4x \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(1 - 4x) = 4y - 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{4y-1}{1-4x}.$$

④ (i) $y = \left(1 + \frac{1}{x}\right)^3 \Rightarrow \frac{dy}{dx} = 3\left(1 + \frac{1}{x}\right)^2 \cdot \left(\frac{-1}{x^2}\right)$

$$= -\frac{3\left(1 + \frac{1}{x}\right)^2}{x^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -3\left\{\frac{x^2 \cdot 2\left(1 + \frac{1}{x}\right)\left(-\frac{1}{x^2}\right) - \left(1 + \frac{1}{x}\right)^2 \cdot 2x}{x^4}\right\}$$

(ii) $y = \sin x + \cos x$

$$\Rightarrow \frac{dy}{dx} = \cos x - \sin x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\sin x - \cos x.$$