PHU11104

Lecture Notes on

Linear Algebra

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Main Text: Calculus for the Life Sciences by

Bittenger, Brand and Quintanilla

Other Text: Linear Algebra by

Anton and Rorres

Matrices

An $m \times n$ real matrix A is an array of mn real numbers arranged in m rows and n columns,

$$A = \begin{pmatrix} a_{11} & a_{12} \dots & a_{1j} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2j} \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} \dots & a_{ij} \dots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \dots & a_{mj} \dots & a_{mn} \end{pmatrix}.$$

The $1 \times n$ matrix $\begin{pmatrix} a_{i1} & a_{i2} \dots & a_{in} \end{pmatrix}$ is called the i^{th} row

and the $m \times 1$ matrix $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$ is called the j^{th} column.

A is often written just as $A = (a_{ij})$ for short and a_{ij} is called the ij-entry or component. If $m = n$, then A is a square matrix. $m \times n$ is called the size of the matrix.

Examples: (i) $\begin{pmatrix} 1 & -1 \\ -7 & 32 \end{pmatrix}$ is a 2×2 matrix.

(ii) $\begin{pmatrix} 1 & 0 & -5 \\ 6 & 3 & 8 \end{pmatrix}$ is a 2×3 matrix.

(iii) $\begin{pmatrix} 1 & 5 & 0 & 21 \\ 2 & -7 & 1 & 5 \\ 6 & 8 & -1 & 0 \\ -2 & 3 & 10 & 9 \end{pmatrix}$ is a 4×4 matrix.

Definition: Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size are equal if all their corresponding entries are equal i.e. $a_{ij} = b_{ij}$ for all i and j.

Definition: We define addition of matrices of the same size componentwise: if $A = (a_{ij})$ and $B = (b_{ij})$ are $m \times n$ matrices, then $A + B = C$, where $C = (c_{ij})$ with $c_{ij} = a_{ij} + b_{ij}$ for all i and j.

Example: If $A = \begin{pmatrix} 2 & 3 & -5 \\ 0 & 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -5 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$,

then $A + B = \begin{pmatrix} -3 & 3 & -3 \\ 0 & 2 & 7 \end{pmatrix}$.

Definition: Scalar Multiplication: If $A = (a_{ij})$ is an $m \times n$ matrix and α is a real number, then αA is the $m \times n$ matrix given by $\alpha A = (\alpha a_{ij})$.

Example: If $A = \begin{pmatrix} 5 & 1 & 3 \\ 2 & 0 & -4 \\ 1 & 3 & 2 \end{pmatrix}$, then $2A = \begin{pmatrix} 10 & 2 & 6 \\ 4 & 0 & -8 \\ 2 & 6 & 4 \end{pmatrix}$.

Definition: A square $n \times n$ matrix $A = (a_{ij})$ is diagonal if $a_{ij} = 0$ for all $i \neq j$, i.e. all off-diagonal entries are 0.

Example: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is diagonal.

Two particular $n \times n$ diagonal matrices are the $n \times n$ unit matrix $I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix}$ and the $n \times n$ zero matrix

$$0 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Theorem: For matrices A, B, C of the same size we have

(i) $A + B = B + A$

(ii) $(A + B) + C = A + (B + C)$

(iii) $\alpha(A + B) = \alpha A + \alpha B$.

Proof: Exercise (look at components).

Exercise: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 0 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 0 & 2 \\ 1 & -3 & 4 \end{pmatrix}$,

find $A + B, \quad 6A - 3B, \quad 21B - \frac{1}{2}A$.

Definition: Multiplication of matrices is defined as follows:

if $A = (a_{ij})$ is $m \times p$ and $B = (b_{jk})$ is $p \times n$, then we define the product $C = AB$ to be the $m \times n$ matrix (c_{ik}), where $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{ip}b_{pk}$.

Note that the number of columns on the left must equal the number of rows on the right.

Examples: (i) If $A = \begin{pmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \end{pmatrix}$ is 2×3 and $B = \begin{pmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{pmatrix}$ is 3×2, then AB is defined and $AB = \begin{pmatrix} 15 & 15 \\ 4 & 12 \end{pmatrix}$

is 2×2.

(ii) If $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{pmatrix}$ is 2×3 and $B = \begin{pmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 4 & 5 & 2 \end{pmatrix}$ is 3×4, then AB is defined and $AB = \begin{pmatrix} 12 & 15 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{pmatrix}$ is (2×4).

(iii) If $A = \begin{pmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \\ -1 & 4 & -2 \end{pmatrix}$ is 3×3 and $B = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ is 3×1, then AB is defined and $AB = \begin{pmatrix} 15 \\ 4 \\ -11 \end{pmatrix}$ is 3×1.

Note that BA need not be defined even if AB is and if it is defined, $BA \neq AB$ in general.

Theorem: If A is an $m \times n$ matrix and B, C are $n \times p$ matrices and $\alpha \in \mathbb{R}$, then

(i) $A(B + C) = AB + AC$

(ii) $A(\alpha B) = \alpha(AB)$.

Proof: Exercise (again look at components.)

Exercise: (i) If $A = \begin{pmatrix} 2 & -5 & 1 \\ -1 & 3 & 1 \\ 2 & -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

find AB.

(ii) If $A = \begin{pmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{pmatrix}$

find $AB, BA, 3A - 2B$.

Systems of Linear Equations

The system of m equations in n variables (unknowns) is a set of equations of the form

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m.$$

This system can be written in matrix form as

$$\begin{pmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \text{ or simply } Ax = b.$$

A is called the coefficient matrix and

$$[A|b] = \begin{pmatrix} a_{11} & a_{12} \ldots & a_{1n} & | & b_1 \\ a_{21} & a_{22} \ldots & a_{2n} & | & b_2 \\ \vdots & \vdots & \vdots & | & \vdots \\ a_{m1} & a_{m2} \ldots & a_{mn} & | & b_m \end{pmatrix} \text{ is called the augmented}$$

matrix.

The augmented matrix is essentially the system of equations omitting the variables and the = signs.

Recall the case of two variables:

Examples: (i)

$$2x + 3y = -3$$
$$x - 2y = 2$$

Geometrically we have two non-parallel lines and hence have a unique solution.

(ii)

$$2x + 3y = -3$$
$$2x + 3y = 1$$

Geometrically we have two different parallel lines and hence have no solution.

(iii)

$$2x + 3y = -3$$
$$4x + 6y = -6$$

Geometrically we have two parallel lines which are the same line and hence infinitely many solutions.

The solution set of the system is the set of all $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ that satisfies all the equations. We can perform the following three operations on the equations without altering the solution set:

(i) Interchange any two equations;

(ii) multiply any equation by a constant;

(iii) add or subtract any multiple of one equation to or from another equation.

We solve the system by replacing it with a new system which has the same solution set but which is easier to solve. We do this by using the above operations.

When these operations are applied to the augmented matrix they are called elementary row operations. There are three types of elementary row operations:

(i) R_{ij}; interchange the i^{th} and j^{th} rows

(ii) αR_i; multiply the i^{th} row by α

(iii) $R_i + \alpha R_j$; add α times i^{th} row to j^{th} row.

Note that none of these elementary row operations will change the solution set of the system.

Example: Solve the system of linear equations

$$x_1 + x_2 + 2x_3 = 9$$

$$2x_1 + 4x_2 - 3x_3 = 1$$

$$3x_1 + 6x_2 - 5x_3 = 0$$

We perform suitable operations on the augmented matrix.

$$\begin{pmatrix} 1 & 1 & 2 & \vdots & 9 \\ 2 & 4 & -3 & \vdots & 1 \\ 3 & 6 & -5 & \vdots & 0 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 2 & \vdots & 9 \\ 0 & 2 & -7 & \vdots & -17 \\ 3 & 6 & -5 & \vdots & 0 \end{pmatrix} \xrightarrow{R_3 - 3R_1}$$

$$\begin{pmatrix} 1 & 1 & 2 & \vdots & 9 \\ 0 & 2 & -7 & \vdots & -17 \\ 0 & 3 & -11 & \vdots & -27 \end{pmatrix} \xrightarrow{R_2 - R_3} \begin{pmatrix} 1 & 1 & 2 & \vdots & 9 \\ 0 & -1 & 4 & \vdots & 10 \\ 0 & 3 & -11 & \vdots & -27 \end{pmatrix} \xrightarrow{(-1)R_2}$$

$$\begin{pmatrix} 1 & 1 & 2 & \vdots & 9 \\ 0 & 1 & -4 & \vdots & -10 \\ 0 & 3 & -11 & \vdots & -27 \end{pmatrix} \xrightarrow{R_3 - 3R_2} \begin{pmatrix} 1 & 1 & 2 & \vdots & 9 \\ 0 & 1 & -4 & \vdots & -10 \\ 0 & 0 & 1 & \vdots & 3 \end{pmatrix} \xrightarrow{R_1 - R_2}$$

$$\begin{pmatrix} 1 & 0 & 6 & : & 19 \\ 0 & 1 & -4 & : & -10 \\ 0 & 0 & 1 & : & 3 \end{pmatrix} \xrightarrow{R_1-6R_3} \begin{pmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & -4 & : & -10 \\ 0 & 0 & 1 & : & 3 \end{pmatrix} \xrightarrow{R_2+4R_3}$$

$\begin{pmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{pmatrix}$. The system has been reduced to

$$x_1 = 1$$
$$x_2 = 2$$
$$x_3 = 3.$$

This is an example of a general method, called row reduction or Gaussian elimination, to solve such systems.

Definition: A matrix is in row-echelon form if

(i) every non-zero row begins with 1, called a leading 1;

(ii) any zero rows are at the bottom;

(iii) in any two successive non-zero rows the leading 1 in the lower row occurs further to the right than the leading 1 in the higher row.

The variables associated with the leading 1s are called leading variables.

The matrix is in reduced row-echelon form if, in addition, each column that contains a leading 1 has 0 everywhere else.

Examples:

$$\begin{pmatrix} 1 & 4 & 6 & 7 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 1 & 5 \end{pmatrix} \text{ is in row-echelon form.}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ are all in}$$

reduced row-echelon form.

Example: Write the matrix

$$\begin{pmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & 1 \end{pmatrix} \text{ in reduced row-echelon form.}$$

$$\begin{pmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & 1 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & 1 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1}$$

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & 1 \end{pmatrix} \xrightarrow{R_3-2R_1} \begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -27 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_2}$$

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -27 \end{pmatrix} \xrightarrow{R_3-5R_2} \begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 3 \end{pmatrix} \xrightarrow{2R_3}$$

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{pmatrix}$$. This is row-echelon form.

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{pmatrix} \xrightarrow{R_1+5R_2} \begin{pmatrix} 1 & 2 & 0 & 3 & -\frac{23}{2} & -16 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{pmatrix} \xrightarrow{R_1+\frac{23}{2}R_3}$$

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 & 53 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{pmatrix} \xrightarrow{R_2+\frac{7}{2}R_3} \begin{pmatrix} 1 & 2 & 0 & 3 & 0 & 53 \\ 0 & 0 & 1 & 0 & 0 & 15 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{pmatrix}.$$

This is reduced row-echelon form.

There are many ways in which a given matrix can be reduced to reduced row-echelon form but it can be shown that the result is always the same i.e. the reduced row-echelon form is unique. We solve a system of linear equations by reducing the augmented matrix to reduced row-echelon form. This process is called row reduction or Gaussian elimination.

A system of linear equations is homogeneous if all the numbers on the right-hand side are 0 i.e. $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

A homogeneous system always has at least one solution, namely $x_1 = 0, x_2 = 0, \cdots, x_n = 0$. If it has more variables than equations i.e.$m < n$, then not all variables can be leading variables and so we have infinitely many solutions.

If $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, we say the system is inhomogeneous.

Then we can have no solution if a row in the reduced row-echelon form looks like $(\,0 \quad 0 \quad \cdots \quad 0 \quad | \quad \neq 0\,)$, a unique solution if all variables are leading variables or infinitely many solutions if not all variables are leading variables.

Example: Using row reduction solve the system of linear equations

$$x_1 + x_2 + 2x_3 + 3x_4 = 13$$

$$x_1 - 2x_2 + x_3 + x_4 = 8$$

$$3x_1 + x_2 + x_3 - x_4 = 1.$$

$$\begin{pmatrix} 1 & 1 & 2 & 3 & : & 13 \\ 1 & -2 & 1 & 1 & : & 8 \\ 3 & 1 & 1 & -1 & : & 1 \end{pmatrix} \xrightarrow{R_2-R_1} \begin{pmatrix} 1 & 1 & 2 & 3 & : & 13 \\ 0 & -3 & -1 & -2 & : & -5 \\ 3 & 1 & 1 & -1 & : & 1 \end{pmatrix} \xrightarrow{R_3-3R_1}$$

$$\begin{pmatrix} 1 & 1 & 2 & 3 & : & 13 \\ 0 & 3 & 1 & 2 & : & 5 \\ 0 & -2 & -5 & -10 & : & -38 \end{pmatrix} \xrightarrow{(-1)R_3} \begin{pmatrix} 1 & 1 & 2 & 3 & : & 13 \\ 0 & 3 & 1 & 2 & : & 5 \\ 0 & 2 & 5 & 10 & : & 38 \end{pmatrix} \xrightarrow{R_2-R_3}$$

$$\begin{pmatrix} 1 & 1 & 2 & 3 & : & 13 \\ 0 & 1 & -4 & -8 & : & -33 \\ 0 & 2 & 5 & 10 & : & 38 \end{pmatrix} \xrightarrow{R_3-2R_2} \begin{pmatrix} 1 & 1 & 2 & 3 & : & 13 \\ 0 & 1 & -4 & -8 & : & -33 \\ 0 & 0 & 13 & 26 & : & 104 \end{pmatrix} \xrightarrow{\frac{1}{13}R_3}$$

$$\begin{pmatrix} 1 & 1 & 2 & 3 & \vdots & 13 \\ 0 & 1 & -4 & -8 & \vdots & -33 \\ 0 & 0 & 1 & 2 & \vdots & 8 \end{pmatrix} \xrightarrow{R_1-R_2} \begin{pmatrix} 1 & 0 & 6 & 11 & \vdots & 46 \\ 0 & 1 & -4 & -8 & \vdots & -33 \\ 0 & 0 & 1 & 2 & \vdots & 8 \end{pmatrix} \xrightarrow{R_2+4R_3}$$

$$\begin{pmatrix} 1 & 0 & 6 & 11 & \vdots & 46 \\ 0 & 1 & 0 & 0 & \vdots & -1 \\ 0 & 0 & 1 & 2 & \vdots & 8 \end{pmatrix} \xrightarrow{R_1-6R_3} \begin{pmatrix} 1 & 0 & 0 & -1 & \vdots & -2 \\ 0 & 1 & 0 & 0 & \vdots & -1 \\ 0 & 0 & 1 & 2 & \vdots & 8 \end{pmatrix}.$$

We have reduced the system to the form

$$x_1 - x_4 = -2$$

$$x_2 = -1$$

$$x_3 + 2x_4 = 8$$

x_1, x_2 and x_3 are called leading variables. x_4 is called a free variable, it can have any value r, say. Then $x_1 = -2 + r, x_2 = -1, x_3 = 8 - 2r, x_4 = r$ is the solution set of the system. In vector form the solution set is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 8 \\ 0 \end{pmatrix} + r \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

Example: Using row reduction solve the system of linear equations

$$x_1 + 6x_2 + 4x_3 = 1$$

$$2x_1 + 4x_2 - x_3 = 0$$

$$-x_1 + 2x_2 + 5x_3 = 3.$$

$$\begin{pmatrix} 1 & 6 & 4 & \vdots & 1 \\ 2 & 4 & -1 & \vdots & 0 \\ -1 & 2 & 5 & \vdots & 3 \end{pmatrix} \xrightarrow{R_3+R_1} \begin{pmatrix} 1 & 6 & 4 & \vdots & 1 \\ 2 & 4 & -1 & \vdots & 0 \\ 0 & 8 & 9 & \vdots & 4 \end{pmatrix} \xrightarrow{R_2-2R_1}$$

$$\begin{pmatrix} 1 & 6 & 4 & \vdots & 1 \\ 0 & -8 & -9 & \vdots & -2 \\ 0 & 8 & 9 & \vdots & 4 \end{pmatrix} \xrightarrow{R_3+R_2} \begin{pmatrix} 1 & 6 & 4 & \vdots & 1 \\ 0 & -8 & -9 & \vdots & -2 \\ 0 & 0 & 0 & \vdots & 2 \end{pmatrix}.$$

The bottom line is not true for any values of the variables, so the system has no solution.