PHU11104

Lecture Notes on

Calculus

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Main Text: Calculus for the Life Sciences by

Bittenger, Brand and Quintanilla

Other Text: Calculus by

Anton, Bivens and Davis

Real Number System

$\mathbb{N} = \{1, 2, 3, \cdots\}$ is called the set of **n**atural numbers or positive integers.

$\mathbb{Z} = \{\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\}$ is called the set of integers.

$\mathbb{Z}$ comes from the German word **z**ahl, meaning number.

$\mathbb{Q} = \{\frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0\}$ is called the set of rational numbers or fractions or **q**uotients.

Any fraction can be written as $\frac{a}{b}$ where a and b have no common factors except 1 e.g. $\frac{6}{8} = \frac{3}{4}$.

Consider $\sqrt{a}$ where a is a positive rational. Some numbers of this type are rational e.g. $\sqrt{\frac{9}{16}} = \frac{3}{4}$, but some are not.

Theorem: $\sqrt{2}$ is not rational.

Proof: Suppose $\sqrt{2} = \frac{a}{b}$ where a and b have no common factors except 1. Then $2 = \frac{a^2}{b^2}$ so $a^2 = 2b^2$. Now $2b^2$ is divisible by 2, so a^2 is divisible by 2 and hence a is divisible by 2. Suppose $a = 2c$. Then $a^2 = 4c^2$, so $2b^2 = 4c^2$ or $b^2 = 2c^2$. Now $2c^2$ is divisible by 2, so b^2 is divisible by 2 and hence b is divisible by 2. But now we have both a and b are divisible by 2, a contradiction to our assumption. We deduce that $\sqrt{2}$ is not rational.

Non-rational numbers of the type $\sqrt{a}, \sqrt{a}+\sqrt{b}, \sqrt{a}-\sqrt{b}$, and quotients of them, are called surds. There are other non-rational numbers which cannot be written as surds e.g. π, e. These are called transcendental numbers. All the surd numbers together with the transcendental numbers constitute the set of irrational numbers.

Finally $\mathbb{R} = \mathbb{Q} \cup \{irrationals\}$ is the set of **r**eal numbers. We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Real numbers can be represented by the points on a line, the real number line:

Recall how we add, subtract and multiply fractions:

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}, \quad \frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}, \quad \frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}.$$

We have $\frac{a}{b}\times\frac{1}{\frac{a}{b}}=1$ and $\frac{a}{b}\times\frac{b}{a}=1$, so $\frac{1}{\frac{a}{b}}=\frac{b}{a}$ and then division is given by $\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\times\frac{1}{\frac{c}{d}}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$.

In practice we usually get the common denominator.

Example: $\frac{5}{6}+\frac{4}{9}=\frac{15+8}{18}=\frac{23}{18}=1\frac{5}{18}$, a mixed number.

Exercise: Simplify the following

(i) $\frac{3}{4}-\frac{1}{3}+\frac{2}{5}$

(ii) $\frac{2}{3}+\frac{3}{5}\times 1\frac{1}{11}$

(iii) $7\frac{3}{4}\div 3\frac{2}{3}$

(iv) $5\frac{1}{5}\times 7\frac{2}{3}$.

Rational Indices or Powers

Definition: Let $n \in \mathbb{N}$ and $a \in \mathbb{R}, a > 0$.

$a^n = a \times a \times a \times \cdots \times a$ (n times)

$a^{-n} = \frac{1}{a^n}$.

Then $a^m \times a^n = (a \times a \times a \times \cdots \times a) \times (a \times a \times a \times \cdots \times a) = a^{m+n}$

and for $m > n$, $\frac{a^m}{a^n} = \frac{a \times a \times a \times \cdots \times a}{a \times a \times a \times \cdots \times a} = a^{m-n}$.

Also $(a^m)^n = a^m \times a^m \times a^m \times \cdots \times a^m = a^{mn}$.

Now $\frac{a^n}{a^n} = 1$, so if we agree that $a^0 = 1$, then the above formula holds for $m = n$ also.

If $m < n$, then $\frac{a^m}{a^n} = \frac{1}{a^{n-m}} = a^{m-n}$ again.

We define $a^{\frac{1}{n}}$ to be the positve number whose n^{th} power is a i.e. $a^{\frac{1}{n}} = x$ where $x^n = a$. $a^{\frac{1}{n}}$ is called the n^{th} root of a.

Finally we define $a^{\frac{m}{n}}$ to be $(a^{\frac{1}{n}})^m$ or $(a^{\frac{1}{m}})^n$.

Examples: (i) $(36)^{-\frac{3}{2}} = \frac{1}{(36)^{\frac{3}{2}}} = \frac{1}{(6^2)^{\frac{3}{2}}} = \frac{1}{6^3} = \frac{1}{216}$.

(ii) $(2\frac{1}{4})^{1\frac{1}{2}} = (\frac{9}{4})^{\frac{3}{2}} = ((\frac{3}{2})^2)^{\frac{3}{2}} = (\frac{3}{2})^3 = \frac{27}{8}$.

Exercise: Simplify $81^{\frac{3}{4}}$, $\quad (\frac{27}{125})^{-\frac{2}{3}}$.

Question: What is a^x if x is not rational? $(a > 0.)$

Logarithms

Definition: Suppose that a, x and y are rational numbers. We say that $\log_a x = y$ if and only if $a^y = x$.

Examples: (i) $\log_5 25 = 2$ since $5^2 = 25$.

(ii) Evaluate $\log_{\frac{1}{2}} 8$. Let $y = \log_{\frac{1}{2}} 8$. Then $(\frac{1}{2})^y = 8$, so $2^{-y} = 2^3$ and hence $y = -3$.

Properties of logarithms: (i) $\log_a(xy) = \log_a x + \log_a y$.

Proof: Let $c = \log_a x$ and $d = \log_a y$. Then $a^c = x$ and $a^d = y$, so $xy = a^c a^d = a^{c+d}$. Hence $\log_a(xy) = c + d = \log_a x + \log_a y$.

(ii) $\log_a(\frac{x}{y}) = \log_a x - \log_a y$.

Proof: Exercise.

(iii) $\log_a(x^y) = y \log_a x$.

Proof: Let $c = \log_a x$, so $a^c = x$ and then $x^y = (a^c)^y = a^{cy}$. Therefore $\log_a(x^y) = \log_a(a^{cy}) = cy = yc = y \log_a x$.

(iv) $\log_b x = \frac{\log_a x}{\log_a b}$.(Change of base.)

Proof: Let $y = \log_b x$, so that $b^y = x$ and then $\log_a(b^y) = \log_a x$. This means that $y \log_a b = \log_a x$ and so $\log_b x = y = \frac{\log_a x}{\log_a b}$.

Examples: (i) $\log_4 64 = \log_4(4^3) = 3\log_4 4 = 3$.

(ii) $\log_3(\frac{1}{3}) = \log_3(3^{-1}) = -\log_3 3 = -1$.

(iii) $(\log_2 5)(\log_5 8) = (\log_2 5)(\frac{\log_2 8}{\log_2 5}) = \log_2(2^3) = 3\log_2 2 = 3$.

(iv) If $\log_3 b + \log_9 b = \frac{3}{4}$, find b.

$\frac{3}{4} = \log_3 b + \log_9 b = \log_3 b + \frac{\log_3 b}{\log_3 9} = \log_3 b + \frac{\log_3 b}{2} = \frac{3}{2}\log_3 b$.

Therefore $\log_3 b = \frac{1}{2}$, so $b = 3^{\frac{1}{2}} = \sqrt{3}$.

(v) Solve for x if $\log_2 x = 2 + \log_2 3$.

$\log_2 x = 2+\log_2 3$, so $\log_2 x - \log_2 3 = 2$ and hence $\log_2(\frac{x}{3}) = 2$.

Therefore $\frac{x}{3} = 2^2 = 4$ and so $x = 12$.

(vi) Solve for x if $2\log_3 x - \log_3(x+6) = 1$.

$2\log_3 x - \log_3(x+6) = 1$, so $\log_3(x^2) - \log_3(x+6) = 1$ and hence $\log_3(\frac{x^2}{x+6}) = 1$. Therefore $\frac{x^2}{x+6} = 3$, so $x^2 = 3x + 18$ or $x^2 - 3x - 18 = 0$. We have $(x-6)(x+3) = 0$ and so $x = 6$.

Exercise: (i) Solve for b if $\log_b 3 + \log_b 27 = 2$.

(ii) Solve for x if $\log_{10}(6x-1) - \log_{10} 2 = 1$.

(iii) Solve for x if $2\log_{10} x - \log_{10}(20-x) = 1$.

Functions of a Real Variable

A real variable x is a representative of the set of real numbers $\mathbb{R}$. Its value ranges over the whole set $\mathbb{R}$.

Definition: $\mathbb{R} \times \mathbb{R} = \{(x, y)|x, y \in \mathbb{R}\}$, also written as $\mathbb{R}^2$. It is the set of ordered pairs of real numbers.

Definition: Any subset of $\mathbb{R} \times \mathbb{R}$ is called a relation from $\mathbb{R}$ to $\mathbb{R}$. e.g. $r = \{(1, 2), (3, 3), (3, 5)\}$ is a relation from $\mathbb{R}$ to $\mathbb{R}$.

Definition: A function from $\mathbb{R}$ to $\mathbb{R}$ is a relation from $\mathbb{R}$ to $\mathbb{R}$ such that no two pairs have the same first element. r above is not a function.

We can represent a function diagrammatically by a "Papy-gram":

We write $(x, y) \in f$ (the passive viewpoint) or $y = f(x)$ (the active viewpoint). We also write a function as

$$\text{(origin) } x \longrightarrow f \longrightarrow y \text{ (image).}$$

The **domain** of f is the set of origins and the **range** of f is the set of images.

Note: Any rule that gives a unique image for x defines a function.

Definition: The graph of f is $\{(x, f(x)) | x \in \text{domain } f\}$, and is represented geometrically on a pair of perpendicular axes (x and y axes.)

Examples: (i) $f : \mathbb{R} \to \mathbb{R} : x \mapsto x$ or $f(x) = x$ or $y = x$.

The domain is $\mathbb{R}$ and the range is $\mathbb{R}$.

(ii) $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$ or $f(x) = x^2$ or $y = x^2$.

The domain is $\mathbb{R}$ and the range is $\mathbb{R}^+ \cup \{0\}$.

Sometimes the set on the right is bigger than the range; it is called the codomain, in general. In (ii) the codomain is $\mathbb{R}$.

(iii) $f : \mathbb{R} \to \mathbb{R} : x \mapsto 2x^2 - 5$ or $f(x) = 2x^2 - 5$ or $y = 2x^2 - 5$.

The domain is $\mathbb{R}$ and the range is $\{y | y \geq -5\}$.

(iv) $f : \mathbb{R} - \{0\} \to \mathbb{R} : x \mapsto \frac{1}{x}$ or $f(x) = \frac{1}{x}$ or $y = \frac{1}{x}$.

The domain is $\mathbb{R} - \{0\}$ and the range is $\mathbb{R} - \{0\}$.

(v) $f : \mathbb{R} \to \mathbb{R} : x \mapsto c$ or $f(x) = c$ or $y = c$, where c is a constant i.e. a fixed number.

The domain is $\mathbb{R}$ and the range is c.

Suppose that a, b, c, d are constants.

Any function of the form $f(x) = ax + b$ is called linear.

Any function of the form $f(x) = ax^2 + bx + c$ is called quadratic.

Any function of the form $f(x) = ax^3 + bx^2 + cx + d$ is called cubic.

Similarly we have quadric, quintic etc. functions. In general, a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0,$$

where $a_n, a_{n-1}, \cdots, a_1, a_0$ are constants, is called a polynomial function.

New Functions from Old

Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions.

We define $f + g : \mathbb{R} \to \mathbb{R}$ by $(f + g)(x) = f(x) + g(x)$ (sum),

$f - g : \mathbb{R} \to \mathbb{R}$ by $(f - g)(x) = f(x) - g(x)$ (difference),

$fg : \mathbb{R} \to \mathbb{R}$ by $(fg)(x) = f(x)g(x)$ (product),

$\frac{f}{g} : \mathbb{R} \to \mathbb{R}$ by $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$ (quotient).

Example: $f(x) = 1 + \sqrt{x - 2}, \quad g(x) = x - 3.$

Then $(f + g)(x) = 1 + \sqrt{x - 2} + x - 3 = \sqrt{x - 2} + x - 2,$

$(f - g)(x) = 1 + \sqrt{x - 2} - (x - 3) = \sqrt{x - 2} - x + 4,$

$(fg)(x) = (1 + \sqrt{x - 2})(x - 3)$, and $(\frac{f}{g})(x) = \frac{1+\sqrt{x-2}}{x-3}$.

A quotient function of the form $\frac{f}{g}$ is called a rational function.

There is another way of defining a new function from two given functions, namely by composition:

for $f, g : \mathbb{R} \to \mathbb{R}$ define $fog : \mathbb{R} \to \mathbb{R}$ by $(fog)(x) = f(g(x))$.

Example: $f(x) = x^2, \quad g(x) = x + 1.$

Then $(fog)(x) = f(g(x)) = f(x + 1) = (x + 1)^2$.

Note that $(gof)(x) = g(x^2) = x^2 + 1 \neq (fog)(x)$.

Exercise: If $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$, find $(fog)(x)$ and $(gof)(x)$.

A simple but important function is the modulus function $| \quad |$:

Definition: $| \quad |: \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ is defined by

$$| x |= \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

e.g. $| 3 |= 3, \quad | -3 |= -(-3) = 3$.

Modulus is often called the absolute value. Consider $| x |< 3$. If $x \geq 0$, then $x < 3$, whereas if $x < 0$, then $-x < 3$, so $x > -3$.

Hence $\{x| \mid x |< 3\} = \{x| - 3 < x < 3\}$. This is called the open interval about 0 of length 3, written as $(-3, 3)$.

Similarly $\{x| \mid x |\leq 3\} = \{x| - 3 \leq x \leq 3\}$. This is called the closed interval about 0 of length 3, written as $[-3, 3]$.

Consider $\mid x - 2 \mid < 3$.

If $x - 2 \geq 0$, then $x - 2 < 3$, so $x < 5$, whereas if $x - 2 < 0$, then $-(x - 2) < 3$, so $x - 2 > -3$ or $x > -1$.

Hence $\{x \mid \mid x - 2 \mid < 3\} = \{x \mid -1 < x < 5\}$. This is the open interval about 2 of length 3, written as $(-1, 5)$.

Similarly we get the closed interval about 2 of length 3, written as $[-1, 5]$.

In general $\mid x - a \mid < b$ will give the open interval about a of length b, written as $(a - b, a + b)$.

Inverse Functions

Suppose that $A, B \subseteq \mathbb{R}$.

Definition: If $f : A \longrightarrow B$ is a function then $f^{-1} : B \longrightarrow A$ is defined as the relation $\{(y, x) | (x, y) \in f\}$.

Example: $A = \{1, 2, 3\}, B = \{10, 11\}$

$f = \{(1, 10), (2, 10), (3, 11)\}, \quad f^{-1} = \{(10, 1), (10, 2), (11, 3)\}.$

f^{-1} is not a function since $10 \longrightarrow 1$ and $10 \longrightarrow 2$.

Definition: $f : A \longrightarrow B$ is injective or one to one (1-1) if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, i.e. at most one arrow arriving at each element in the range of f. The example above is not 1-1.

Example: $A = \{1, 2, 3\}, B = \{10, 11, 12, 13\}$

$g = \{(1, 10), (2, 12), (3, 11)\}, \quad g^{-1} = \{(10, 1), (12, 2), (11, 3)\}.$

g is 1-1, but g^{-1} is still not a function on the given codomain of g since 13 is not mapped onto anything by g^{-1}.

Definition: $f : A \longrightarrow B$ is surjective or onto if given any $y \in B$ there exists some $x \in A$ such that $y = f(x)$, i.e. at least one arrow arriving at each element in B or the range of f equals the codomain of f. The example above is not onto.

Definition: $f : A \longrightarrow B$ is bijective if it is both injective and surjective i.e. one arrow leaving everything in the domain and one arrow arriving at everything in the codomain. In this case f^{-1} is a function.

Note: $y = f(x)$ and $f^{-1}(y) = x$

then $f^{-1}(f(x)) = x$ or $(f^{-1} o f)(x) = x$

and $f(f^{-1}(y)) = y$ or $(f o f^{-1})(y) = y$.

We see that f^{-1} is the inverse of f under composition.

If f is given by some algebraic rule we can use algebra to get the rule for f^{-1}.

Example: If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is given by $x \mapsto 3x - 2$, find the rule for f^{-1}.

Let $y = 3x - 2$; then $y + 2 = 3x$ so $x = \frac{y+2}{3}$ and therefore $x = f^{-1}(y) = \frac{y+2}{3}$.

If there is no confusion we usually write $f^{-1}(x) = \frac{x+2}{3}$.

Note that the range of f is all of $\mathbb{R}$ and so is equal to the codomain of f

Example: If $f : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}^+ \cup \{0\}$ is given by $x \mapsto x^2$, find the rule for f^{-1}.

Again note that the range of f is equal to the codomain of f, so f^{-1} exists and is given by $f^{-1}(x) = \sqrt{x}$.

Limits and Continuous Functions

Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$

Informal Definition: If the value of $f(x)$ can be made "as close as we like" to L by taking x sufficiently close to a, but not equal to a, then we say that the limit of $f(x)$, as x approaches a is L, written as $\lim\limits_{x \to a} f(x) = L$.

Examples: (i) $f(x) = x$

$\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} x = 2.$

(ii) $f(x) = x^2$

$\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} x^2 = 3^2 = 9.$

(iii) $f(x) = x^3 + 2x + 5$

$\lim\limits_{x \to 5} f(x) = \lim\limits_{x \to 5} (x^3 + 2x + 5)$

$= 5^3 + 2(5) + 5 = 140.$

(iv) $f(x) = c$, a constant

$\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} c = c.$

Theorems on Limits

(1) $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ and $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.

(2) $\lim_{x\to a}(f(x)g(x)) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$.

(3) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$.

(4) $\lim_{x\to a}(f(x)^{\frac{m}{n}}) = (\lim_{x\to a} f(x))^{\frac{m}{n}}$.

Examples: (i) $\lim_{x\to 5}(x^3+2x+5) = \lim_{x\to 5}(x^3) + \lim_{x\to 5}(2x) + \lim_{x\to 5}(5) = 5^3 + 2(5) + 5 = 140$.

(ii) $\lim_{x\to 4} \frac{3x-2}{x+3} = \frac{\lim_{x\to 4}(3x-2)}{\lim_{x\to 4}(x+3)} = \frac{3(4)-2}{4+3} = \frac{10}{7}$.

(iii) $\lim_{x\to 2} \frac{5x^3-40}{x-3} = \frac{\lim_{x\to 2}(5x^3-40)}{\lim_{x\to 2}(x-3)} = \frac{5(2)^3-40}{2-3} = \frac{0}{-1} = 0$.

(iv) $\lim_{x\to -2}(3x^2-5x+9) = 3(-2)^2 - 5(-2) + 9 = 31$.

(v) $\lim_{x\to 3} \sqrt{\frac{x^2+3}{x+4}} = \sqrt{\lim_{x\to 3} \frac{x^2+3}{x+4}} = \sqrt{\frac{3^2+3}{3+4}} = \sqrt{\frac{12}{7}}$.

(vi) $\lim_{x\to -4} \frac{2x+8}{x^2+x-12} = \frac{2(-4)+8}{(-4)^2+(-4)-12} = \frac{0}{0} = ?$.

We will discuss this type of problem later.

Exercise: Evaluate the following limits:

(i) $\lim_{x\to 3}(x^2-4x+1)$

(ii) $\lim_{x\to 5} \frac{2x-3}{x+4}$

(iii) $\lim_{x\to 2} \sqrt{\frac{x^2-1}{x+2}}$.

What does "as close as we like" mean?

Rigorous Definition: Let $f(x)$ be defined for all x in some open interval containing a, except perhaps at a itself. We say that $\lim_{x \to a} f(x) = L$ if given any interval of length ϵ about L, no matter how small ϵ is, we can find a corresponding interval of some length δ about a such that whenever $a - \delta < x < a + \delta$, except perhaps $x = a$ itself, then $L - \epsilon < f(x) < L + \epsilon$ or, more succinctly, $0 < \mid x - a \mid < \delta$ implies that $\mid f(x) - L \mid < \epsilon$. In this course the informal approach is taken.

Definition: A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a if

(i) $\lim_{x \to a} f(x)$ exists and

(ii) $\lim_{x \to a} f(x) = f(a)$.

This means that the graph of f does not "break" off at a. If f is not continuous at a we say that it is discontinuous at a. f is a continuous function if it is continuous at all points in its domain.

Example: $f(x) = x^2$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 = 2^2 = 4 = f(2)$.

All the functions that we have met so far in this section except the "$\frac{0}{0}$" example are continuous.

Some functions are continuous everywhere except at a finite number of points and at those points have left-hand and right-hand limits. Such functions are called piecewise continuous.

Example:

$$f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

$\lim_{x \to 0^-} f(x) = -1$

$\lim_{x \to 0^+} f(x) = 1$

and f is continuous everywhere else.

Limits of this type are called one-sided limits, We say that there is a jump-discontinuity at 0.

We now consider some quotient functions.

Example: (i) $f(x) = \frac{5x^3+4}{x-3}$

$\lim\limits_{x \to 2} \frac{5x^3+4}{x-3} = \frac{\lim\limits_{x \to 2}(5x^3+4)}{\lim\limits_{x \to 2}(x-3)} = \frac{5(2)^3+4}{2-3} = \frac{44}{-1} = -44$. f is continuous at 2. However, if we consider $\lim\limits_{x \to 3} \frac{5x^3+4}{x-3}$, we see that the function gets bigger and bigger (with a $+$ or $-$sign) as x gets closer to 3. We write $\lim\limits_{x \to 3^-} \frac{5x^3+4}{x-3} = -\infty$ and $\lim\limits_{x \to 3^+} \frac{5x^3+4}{x-3} = +\infty$. $f(x)$ is not defined at $x = 3$ and does not have a limit as $x \to 3$, so can't be continuous at $x = 3$. The graph of $f(x) = \frac{5x^3+4}{x-3}$ is said to have an asymptote at $x = 3$.

(ii) $f(x) = \frac{x^2-4}{x-2}$

$\lim\limits_{x \to 2} \frac{x^2-4}{x-2} =?$ Again $f(x)$ is not defined at $x = 2$ so it can't be continuous at $x = 2$. However it does have a limit as $x \to 2$: $\lim\limits_{x \to 2} \frac{x^2-4}{x-2} = \lim\limits_{x \to 2} \frac{(x-2)(x+2)}{x-2} = \lim\limits_{x \to 2}(x+2) = 2 + 2 = 4$. We can divide above and below by $x - 2$ since in the definition of the limit we are interested in every point about 2 except the point

2 itself and so $x - 2 \neq 0$ for all the values of x considered. The limit is defined like this so that functions of this type do have limits.

If we define f at 2 to be 4, i.e.

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

then f is continuous at 2.

(iii) $f(x) = \frac{x^2-6x+9}{x-3}$

$\lim\limits_{x \to 3} \frac{x^2-6x+9}{x-3} = \lim\limits_{x \to 3} \frac{(x-3)(x-3)}{x-3} = \lim\limits_{x \to 3}(x-3) = 3-3 = 0.$

(iv) $f(x) = \frac{x^2-9}{x^2-5x+6}$

$\lim\limits_{x \to 3} \frac{x^2-9}{x^2-5x+6} = \lim\limits_{x \to 3} \frac{(x-3)(x+3)}{(x-3)(x-2)} = \lim\limits_{x \to 3} \frac{x+3}{x-2} = \frac{3+3}{3-2} = 6.$

Differentiation

Recall the formula for the slope of a line:

Slope $= m = \tan\alpha$

$= \frac{y_2 - y_1}{x_2 - x_1}$, where

$(x_1, y_1), (x_2, y_2)$ are

any two points on the line.

We wish to define the slope of any continuous curve at any point $p(x, y)$ on the curve. Let the curve be the graph of the continuous function $y = f(x)$.

Slope of the chord PQ is $\frac{f(x+h) - f(x)}{h} = \frac{\Delta y}{\Delta x}$. Intuitively, as $h \to 0, Q \to P$ and the chord $PQ \to$ the tangent to the curve at $P(x, y)$. We define the slope of the curve at $P(x, y)$ to be $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ and the line through $P(x, y)$

with this slope is called the tangent to the curve at $P(x, y)$. This limit is called the derivative of $f(x)$ at $P(x, y)$ and is denoted by $f'(x)$ or $\frac{dy}{dx}$. The process pf finding derivatives is called differentiation.

Differentiation from first principles

Example: (i) $f(x) = c$, a constant

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$

(ii) $f(x) = x$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h-x)}{h}$

$= \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1.$

(iii) $f(x) = x^2$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$

$= \lim_{h\to 0} \frac{(x^2+2xh+h^2-x^2)}{h} = \lim_{h\to 0} \frac{(2xh+h^2)}{h}$

$= \lim_{h\to 0}(2x+h) = 2x.$

In general it can be shown, using the binomial theorem, that

if $f(x) = x^n, n \in \mathbb{N}$, then $f'(x) = nx^{n-1}$.

(iv) $f(x) = \sqrt{x}$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} =$
$\lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})} = \frac{1}{2\sqrt{x}} =$
$\frac{1}{2}x^{-\frac{1}{2}}.$

(v) $f(x) = 2x^2 - 3x + 5$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(2(x+h)^2-3(x+h)+5)-(2x^2-3x+5)}{h}$

$= \lim_{h\to 0} \frac{(4xh+h^2-3h)}{h} == \lim_{h\to 0}(4x+h-3) = 4x-3.$

Theorems (Rules) for Derivatives

(1) Sum and difference rule

$(f+g)'(x) = f'(x) + g'(x)$ and $(f-g)'(x) = f'(x) - g'(x)$.

(2) Product rule

$(fg)'(x) = f(x)g'(x) + g(x)f'(x)$.

(3) Quotient rule

$(\frac{f}{g})'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$.

(4) Chain rule

$(fog)'(x) = f'(g(x))g'(x)$.

In the other notation these rules are expressed as

$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}, \quad \frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$,

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$,

$\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$,

$\frac{d}{dx}(u(v(x)) = \frac{du}{dv}\frac{dv}{dx}$, where $u = f(x)$ and $v = g(x)$.

Examples: (i) $y = 3x^2 - 7x + 4, \quad \frac{dy}{dx} = 6x - 7$.

(ii) $y = (4x^2 - 1)(7x^3 + x), \quad \frac{dy}{dx} = (4x^2 - 1)(21x^2 + 1)$

$+ (7x^3 + x)(8x)$.

(iii) $y = \frac{x^2-1}{x^4+1}, \quad \frac{dy}{dx} = \frac{(x^4+1)(2x) - (x^2-1)(4x^3)}{(x^4+1)^2}$.

(iv) $y = (x^2 - x + 1)^{23}, \quad \frac{dy}{dx} = 23(x^2 - x + 1)^{22}(2x - 1)$.

Implicit Differentiation

If the function y is not given explicitly by a formula we say that y is an implicit function of x.

Example: Find $\frac{dy}{dx}$ if $xy = 1$.

To find $\frac{dy}{dx}$ we differentiate across with respect to x, remembering that y is an implicit function of x. We get $x\frac{dy}{dx} + 1.y = 0$, so $x\frac{dy}{dx} = -y$ giving $\frac{dy}{dx} = -\frac{y}{x}$. Of course, in this case, we can write y as an explicit function of x as $y = \frac{1}{x}$ and differentiate accordingly, but usually this is not possible.

(ii) Find $\frac{dy}{dx}$ if $x^2 + y^2 = 1$.

Differentiating across with respect to x gives $2x + 2y\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{x}{y}$.

(iii) Find $\frac{dy}{dx}$ if $x^3 + y^3 = 3xy$.

Differentiating across with respect to x gives $3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$, so $\frac{dy}{dx}(3y^2 - 3x) = 3y - 3x^2$ and hence $\frac{dy}{dx} = \frac{y-x^2}{y^2-x}$.

Note that we can differentiate powers x^n and $x^{-n} = \frac{1}{x^n}$ by rule. What about $y = x^{\frac{m}{n}}$? $y = x^{\frac{m}{n}} \Rightarrow y^n = x^m$, so $ny^{n-1}\frac{dy}{dx} = mx^{m-1}$, giving $\frac{dy}{dx} = \frac{m}{n}\frac{x^{m-1}}{y^{n-1}} = \frac{m}{n}x^{m-1}y^{-n+1} = \frac{m}{n}x^{m-1}(x^{\frac{m}{n}})^{-n+1} = \frac{m}{n}x^{m-1}x^{-m+\frac{m}{n}} = \frac{m}{n}x^{\frac{m}{n}-1}$.

Trigonometrc Functions

Geometrically an angle A is defined to be a rotation in the anti-clockwise direction. The full rotation is divided into 360 equal rotations, called degrees and written 360^0.

Consider the acute angle A ($< 90^0$) in a right-angled triangle:

Definition: $\cos A = \frac{a}{c}, \quad \sin A = \frac{b}{c}, \quad \tan A = \frac{b}{a}.$

This definition is independent of the right-angled triangle containing A since, by similar triangles, we have

$\frac{a}{c} = \frac{d}{f}$

$\frac{b}{c} = \frac{e}{f}$

$\frac{b}{a} = \frac{e}{d}.$

We wish to define the trigonometric functions as functions of a real variable. To do this we introduce the idea of **radian measure**. Consider the angle A in the unit circle:

There is a $1-1$ correspondence between the rotation A and the arc-length s that it defines:

$360^0 \longleftrightarrow 2\pi, \quad 180^0 \longleftrightarrow \pi, \quad 90^0 \longleftrightarrow \frac{\pi}{2}$ etc.

The arc-length coresponding to an angle is a real number and is called the radian measure of the angle. When we write an angle in radians we usually denote it by a Greek letter θ etc.

We have only defined the trigonometrc functions of acute angles. Using the unit circle we can define them for any angle:

$\cos\theta = x$

$\sin\theta = y$

$\tan\theta = \frac{y}{x}, \quad x \neq 0.$

We also have the reciprocal functions:

$\sec\theta = \frac{1}{\cos\theta}, \quad \cos\theta \neq 0$

$\csc\theta = \frac{1}{\sin\theta}, \quad \sin\theta \neq 0$

$\cot\theta = \frac{1}{\tan\theta}, \quad \tan\theta \neq 0$

We now consider the values of the trigonometrc functions in the four **quadrants:**

First quadrant

$x > 0, y > 0$, so

$\cos\theta > 0$

$\sin\theta > 0$

$\tan\theta > 0$

Second quadrant

$x < 0, y > 0$, so

$\cos\theta < 0$

$\sin\theta > 0$

$\tan\theta < 0$

Third quadrant

$x < 0, y < 0$, so

$\cos\theta < 0$

$\sin\theta < 0$

$\tan\theta > 0$

Fourth quadrant

$x > 0, y < 0$, so

$\cos\theta > 0$

$\sin\theta < 0$

$\tan\theta < 0$

We	$\sin > 0$	$\text{all} > 0$
get	$\tan > 0$	$\cos > 0$

Using the above definitions we have the idea of **related angles,** where we relate the trigonometrc functions of angles in the higher quadrants to an angle in the first quadrant i.e. an acute angle:

Second quadrant

$\cos\theta = -x = -\cos(\pi - \theta)$

$\sin\theta = y = \sin(\pi - \theta)$

$\tan\theta = \frac{y}{-x} = -\frac{y}{x} = -\tan(\pi - \theta)$

Third quadrant

$\cos\theta = -x = -\cos(\theta - \pi)$

$\sin\theta = -y = -\sin(\theta - \pi)$

$\tan\theta = \frac{-y}{-x} = \frac{y}{x} = \tan(\theta - \pi)$

Fourth quadrant

$\cos\theta = x = \cos(2\pi - \theta)$

$\sin\theta = -y = -\sin(2\pi - \theta)$

$\tan\theta = \frac{-y}{x} = -\frac{y}{x} = -\tan(2\pi - \theta)$

Examples: $\sin\frac{2\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$,

$\cos\frac{7\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$,

$\tan\frac{7\pi}{4} = -\tan\frac{\pi}{4} = -1$.

There are some well-known **trigonometric identities:** Since $x^2+y^2 = 1$ we get $\cos^2\theta+\sin^2\theta = 1$. Dividing across by $\cos^2\theta$ we get $1+\tan^2\theta = \sec^2\theta$ and dividing across by $\sin^2\theta$ we get $1+\cot^2\theta = \csc^2\theta$.

All of the following are found in the tables and can easily be proven:

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$

$\tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B}$

Taking $B = A$ gives the "double angle" formulas:

$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$,

$\sin 2A = 2 \sin A \cos A$ and $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$.

Finally we have sums (differences)$\longleftrightarrow$ products:

$\sin A + \sin B = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$

$\sin A - \sin B = 2\cos(\frac{A+B}{2})\sin(\frac{A-B}{2})$

$\cos A + \cos B = 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2})$

$\cos A - \cos B = -2\sin(\frac{A+B}{2})\sin(\frac{A-B}{2})$.

Limits of Trigonometric Functions

We see from the graphs of the trigonometrc functions that $f(x) = \cos x, f(x) = \sin x$ are continuous everywhere and that $f(x) = \tan x$ is continuous everywhere except at values $x = n\frac{\pi}{2}, n$ odd, where it is not defined.

Important Limit: $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$.

Proof:

Area triangle AOB $<$ area sector AOB $<$ area triangle COB, so $\frac{1}{2}.1.1.\sin x < \frac{1}{2}.1^2.x < \frac{1}{2}.1.\tan x$ and hence $\sin x < x < \tan x$ or $\cos x < \frac{\sin x}{x} < 1$. Therefore $\lim\limits_{x \to 0} \cos x \leq \lim\limits_{x \to 0} \frac{\sin x}{x} \leq \lim\limits_{x \to 0} 1$ which gives $1 \leq \lim\limits_{x \to 0} \frac{\sin x}{x} \leq 1$ and so $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$.

Note: $\lim\limits_{x\to 0}\frac{x}{\sin x} = \frac{1}{\lim\limits_{x\to 0}\frac{\sin x}{x}} = \frac{1}{1} = 1$

and $\lim\limits_{x\to 0}\frac{\tan x}{x} = \lim\limits_{x\to 0}(\frac{\sin x}{x}\frac{1}{\cos x}) = 1(\frac{1}{1}) = 1.$

Examples: (i) $\lim\limits_{x\to 0}\frac{\sin 3x}{x} = \lim\limits_{x\to 0}(\frac{\sin 3x}{3x}.3) = 1.3 = 3.$

(ii) $\lim\limits_{x\to 0}\frac{\sin 7x}{\sin 4x} = \lim\limits_{x\to 0}(\frac{\sin 7x}{7x}.\frac{4x}{\sin 4x}.\frac{7}{4}) = 1.1.\frac{7}{4}.$

(iii) $\lim\limits_{x\to 0}\frac{1-\cos x}{x} = \lim\limits_{x\to 0}(\frac{1-\cos x}{x}.\frac{1+\cos x}{1+\cos x}) = \lim\limits_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}$

$= \lim\limits_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)} = \lim\limits_{x\to 0}(\frac{\sin x}{x}.\frac{\sin x}{1+\cos x}) = 1.\frac{0}{2} = 0.$

Exercise: Evaluate the following limits:

(i) $\lim\limits_{x\to 0}\frac{\sin 6x}{\sin 8x}$

(ii) $\lim\limits_{x\to 0}\frac{\sin^2 x}{3x^2}$

(iii) $\lim\limits_{x\to 0}\frac{\tan 7x}{\sin 3x}$

(iv) $\lim\limits_{x\to 0}\frac{x}{\cos(\frac{\pi}{2}-x))}.$

Derivatives of Trigonometric Functions

$f(x) = \sin x$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{2\cos(x+\frac{h}{2})\sin\frac{h}{2}}{h}$

$= \lim_{h\to 0}(\frac{2}{2}.\cos(x+\frac{h}{2})\frac{\sin\frac{h}{2}}{\frac{h}{2}}) = 1.\cos x.1 = \cos x.$

Similarly the derivative of $\cos x$ is $-\sin x$ and the derivative of $\tan x$ is $\sec^2 x$.

Now if $y = \sec x = \frac{1}{\cos x}$, then $\frac{dy}{dx} = \frac{\cos x.0-1(-\sin x)}{\cos^2 x} = \frac{1}{\cos x}.\frac{\sin x}{\cos x} = \sec x.\tan x.$

Similarly the derivative of $\csc x$ is $-\csc x.\cot x$ and the derivative of $\cot x$ is $-\csc^2 x$.

Examples: (i) $y = x^2\tan x$; $\quad \frac{dy}{dx} = x^2(\sec^2 x) + \tan x(2x)$

(ii) $y = \frac{\sin x}{1+\cos x}$; $\quad \frac{dy}{dx} = \frac{(1+\cos x)(\cos x)-(\sin x)(-\sin x)}{(1+\cos x)^2} = \frac{2}{(1+\cos x)^2}$

(iii) $y = \sin^2 x = (\sin x)^2$; $\quad \frac{dy}{dx} = 2\sin x.\cos x$

(iv) $y = (1+x^2\cos x)^5$;

$\frac{dy}{dx} = 5(1+x^2\cos x)^4(x^2(-\sin x)+\cos x(2x)).$

(v) Find $\frac{dy}{dx}$ if $\sin(x^2y^2) = x$;

$\sin(x^2y^2) = x$ so $\cos(x^2y^2)[2x.y^2 + x^2.2y.\frac{dy}{dx}] = 1$, and hence $\frac{dy}{dx} = \frac{\frac{1}{\cos(x^2y^2)}-2xy^2}{2x^2y}.$

Integration

Definition: A function $F(x)$ is said to be an antiderivative of $f(x)$ if $F'(x) = f(x)$.

Example: $f(x) = x^2, F(x) = \frac{1}{3}x^3; \quad F'(x) = f(x)$.

Also $G(x) = \frac{1}{3}x^3 + c$ (c any constant) satisfies $G'(x) = f(x)$.

So if an antiderivative exists, then we have infinitely many of them by adding constants.

The process of finding antiderivatives is called (indefinite) integration. If $F'(x) = f(x)$ we write $\int f(x)dx = F(x) + c$.

c is called the constant of integration.

Example: (i) $\int x^2 dx = \frac{1}{3}x^3 + c$ since $\frac{d}{dx}(\frac{1}{3}x^3 + c) = x^2$.

In general $\frac{d}{dx}(\int f(x)dx) = f(x)$.

(ii) $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$ since $\frac{d}{dx}(\frac{1}{n+1}x^{n+1} + c) = x^n$.

(iii)$\int \frac{1}{\sqrt{x}}dx = \int x^{-\frac{1}{2}}dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sqrt{x} + c$.

(iv) $\int \frac{1}{x^5}dx = \int x^{-5}dx = \frac{x^{-4}}{-4} + c = -\frac{1}{4x^4} + c$.

(v) $\int \cos x dx = \sin x + c$, since $\frac{d}{dx}(\sin x) = \cos x$.

(vi) $\int \sec^2 x dx = \tan x + c$.

(vii) $\int \sec x \tan x dx = \sec x + c$.

(viii) $\int tan^2 x dx = \int (\sec^2 x - 1) = \tan x - x + c$.

Rules for Integration

(1) $\int cf(x)dx = c\int f(x)dx$.

Proof: Let $\int f(x)dx = F(x)$. Then $F'(x) = f(x)$, so $cF'(x) = cf(x)$ or $(cF)'(x) = cf(x)$, so $\int cf(x)dx = cF(x) = c\int f(x)dx$.

(2) $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$.

Proof: Let $\int f(x)dx = F(x)$ and $\int g(x)dx = G(x)$. Then $F'(x) = f(x)$ and $G'(x) = g(x)$. Hence $(F+G)'(x) = F'(x)+G'(x) = f(x)+g(x)$ and so $\int (f(x)+g(x))dx = (F+G)(x) = F(x) + G(x) = \int f(x)dx + \int g(x)dx$.

Similarly $\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx$.

(3) To integrate products $\int f(x)g(x)dx$ or quotients $\int \frac{f(x)}{g(x)}dx$ we often use "integration by parts", which we will not consider here.

(4) To integrate compositions of the kind $\int (fog)(x)g'(x)dx$ we use "substitution" which we consider later.

Examples: (i) $\int 4\cos x dx = 4\int \cos x dx = 4\sin x + c$.

(ii) $\int (x + x^2)dx = \int x dx + \int x^2 dx = \frac{1}{2}x^2 + \frac{1}{3}x^3 + c$.

(iii) $\int (14x^6 - 3x^2 + 8x + 1)dx = 2x^7 - x^3 + 4x^2 + x + c$.

(iv) $\int \frac{x^2 - 2x^4}{x^4}dx = \int (x^{-2} - 2)dx = -\frac{1}{x} - 2x + c$.

(v) $\int \frac{\cos x}{\sin^2 x}dx = \int \frac{1}{\sin x}\frac{\cos x}{\sin x}dx = \int \csc x \cot x dx = -\csc x + c.$

(vi) $\int \sin^2 x dx = \frac{1}{2}\int (1-\cos 2x)dx = \frac{1}{2}(x - \frac{1}{2}\sin 2x) + c.$

Suppose that $F'(x) = f(x)$. For any other function $g(x)$ we have $\frac{d}{dx}(F(g(x)) = F'(g(x)).g'(x) = f(g(x)).g'(x)$ (chain rule). This means that $\int f(g(x))g'(x)dx = F(g(x)) + c$.

Letting $u = g(x)$ this takes the form $\int f(u)\frac{du}{dx}dx = F(u)+c = \int f(u)du + c$.

In practice, given an integral of the form $I = \int f(g(x))g'(x)dx$, we let $u = g(x)$ and write $du = g'(x)dx = \frac{du}{dx}dx$ to get $I = \int f(u)du + c = F(u) + c = F(g(x)) + c$, where $F'(x) = f(x)$.

Examples: (i) $I = \int (x^2+1)^{50} 2x dx$; let $u = x^2+1$ so that $\frac{du}{dx} = 2x$ or $du = 2xdx$ and $I = \int u^{50} du = \frac{1}{51}u^{51} + c = \frac{1}{51}(x^2+1)^{51} + c$.

(ii) $I = \int \sin(2x+9)dx = \int \sin(2x+9).1dx$; let $u = 2x+9$ so that $\frac{du}{dx} = 2$ or $du = 2dx$ or $\frac{1}{2}du = 1.dx$ and $I = \frac{1}{2}\int \sin u du = -\frac{1}{2}\cos u + c = -\frac{1}{2}\cos(2x+9) + c$.

(iii) $I = \int \cos 7x dx$; let $u = 7x$ so that $\frac{du}{dx} = 7$ or $du = 7dx$ or $\frac{1}{7}du = dx$ and $I = \frac{1}{7}\int \cos u du = \frac{1}{7}\sin u + c = \frac{1}{7}\sin 7x + c$.

(iv) $I = \int (x-8)^{23} dx = \int (x-8)^{23}.1dx$; let $u = x-8$ so that $du = dx$ and $I = \int u^{23}du = \frac{1}{24}u^{24} + c = \frac{1}{24}(x-8)^{24} + c$.

In general, if we have a power (including a root) of a linear function we let $u =$ the linear function.

(v) $I = \int \frac{dx}{(x-6)^5} = \int (x-6)^{-5}dx$; let $u = x - 6$ so that $du = dx$ and $I = \int u^{-5}du = -\frac{1}{4}u^{-4} + c = -\frac{1}{4}(x-6)^{-4} + c$.

(vi) $I = \int x^4\sqrt{3-5x^5}dx$; let $u = 3 - 5x^5$ so that $du = -25x^4dx$ and $I = -\frac{1}{25}\int \sqrt{u}du = -\frac{2}{75}u^{\frac{3}{2}} + c$

$= -\frac{2}{75}(3-5x^5)^{\frac{3}{2}} + c$.

(vii) $I = \int x^2\sqrt{x-1}dx$; let $u = x - 1$ so that $du = dx$ and

$I = \int (u+1)^2\sqrt{u}du = \int (u^2 + 2u + 1)\sqrt{u}du$

$= \int (u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}})du = \frac{2}{7}u^{\frac{7}{2}} + \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + c$

$= \sqrt{u}(\frac{2}{7}u^3 + \frac{4}{5}u^2 + \frac{2}{3}u) + c$

$= \sqrt{x-1}(\frac{2}{7}(x-1)^3 + \frac{4}{5}(x-1)^2 + \frac{2}{3}(x-1)) + c$.

(viii) $I = \int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$; let $u = \sqrt{x}$ so that $du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{dx}{2\sqrt{x}}$ and $I = 2\int \cos u du = 2\sin u + c = 2\sin\sqrt{x} + c$.

(ix) $I = \int \sin^2 x\cos x dx$; let $u = \sin x$ so that $du = \cos x dx$ and $I = \int u^2du = \frac{1}{3}u^3 + c = \frac{1}{3}\sin^3 x + c$.

(x) $I = \int \sin^3 x dx = \int \sin^2 x \sin x dx = \int (1-\cos^2 x)\sin x dx$; let $u = \cos x$ so that $du = -\sin x dx$ and $I = -\int (1-u^2)du =$

$-(u - \frac{1}{3}u^3) + c = \frac{1}{3}\cos^3 x - \cos x + c.$

As long as we have at least one odd power of sin or cos we can let $u =$ the function such that $\frac{du}{dx}$ is the function with the odd power. If both powers are even we must use a double angle formula

$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$ or $\sin 2x = 2\sin x \cos x$:

(xi) $I = \int \sin^2 x \cos^2 x dx = \frac{1}{4}\int (1 - \cos 2x)(1 + \cos 2x) dx =$
$\frac{1}{4}\int (1 - \cos^2 2x) dx = \frac{1}{4}\int (1 - \frac{1}{2}(1 + \cos 4x)) dx$
$= \frac{1}{4}\int (\frac{1}{2} - \frac{1}{2}\cos 4x) dx = \frac{1}{8}\int (1 - \cos 4x) dx = \frac{1}{8}(x - \frac{1}{4}\sin 4x) + c.$

We could also consider $I = \int (\sin x \cos x)^2 dx = \int (\frac{1}{2}\sin 2x)^2 dx =$
$\frac{1}{4}\int \sin^2 2x dx = \frac{1}{8}\int (1 - \cos 4x) dx = \frac{1}{8}(x - \frac{1}{4}\sin 4x) + c.$

Definite Integrals

Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Let a be any fixed point in the domain of f.

Let $A(x)$ be the area under the curve $y = f(t)$ between a and any point $x > a$. We write $A(x) = \int\limits_a^x f(t)dt$ and call it the definite integral of f from a to x.

$A(x+h) - A(x) \approx f(x)h$, so $\frac{A(x+h)-A(x)}{h} \approx f(x)$ and intuitively $\lim\limits_{h\to 0} \frac{A(x+h)-A(x)}{h} = f(x)$ i.e. $A(x)$ is differentiable and $A'(x) = f(x)$ or $\frac{d}{dx}\int\limits_a^x f(t)dt = f(x)$. This fact is called the Fundamental Theorem of Calculus and can be proven rigorously. Now suppose that g is any antiderivative of f i.e.

$g'(x) = f(x)$. Then $A'(x) = g'(x) = f(x)$, so $(A - g)'(x) = 0$ and $(A - g)(x) = c$, a constant. We have $A(x) = g(x) + c$ for all x. Taking $x = a$ gives $A(a) = g(a) + c$ or $0 = g(a) + c$, so $c = -g(a)$. Therefore $A(x) = g(x) - g(a)$. Now taking $x = b$, for any fixed b gives $A(b) = g(b) - g(a)$ i.e. $\int\limits_a^b f(t)dt = g(b) - g(a)$ i.e. the area under the curve from a to b is $g(b) - g(a)$. So to evaluate the area under the curve $y = f(t)$ or $y = f(x)$ (the name of the variable is now irrevelant) we find any antiderivative $g(x)$ and the area is $g(b)-g(a)$ i.e. $\int\limits_a^b f(t)dt = \int\limits_a^b f(x)dx = g(b) - g(a)$. a and b are called the limits of integration.

Note: (i) $\int\limits_b^a f(x)dx = g(a) - g(b) = -(g(b) - g(a)) = -\int\limits_a^b f(x)dx$.

(ii) If $a < c < b$, then $\int\limits_a^c f(x)dx + \int\limits_c^b f(x)dx = g(c) - g(a) + g(b) - g(c) = g(b) - g(a) = \int\limits_a^b f(x)dx$.

(iii) If $f(x) < 0$ on $[a, b]$, then $\int\limits_a^b f(x)dx < 0$ and we define the area as $-\int\limits_a^b f(x)dx$.

(iv) There are functions that have definite integrals but do not have antiderivatives (see the log function later) so we cannot use the above to calculate areas.

Examples: (i) $\int\limits_1^2 x dx = (\frac{1}{2}x^2)|_1^2 = \{(\frac{1}{2}(2)^2) - (\frac{1}{2}(1)^2)\} = 1\frac{1}{4}$.

(ii) $\int\limits_0^\pi \cos x dx = (\sin x)|_0^\pi = \sin\pi - \sin 0 = 0$. The two halves of the integral cancel out. In such a case the area is defined as $\int\limits_0^{\frac{\pi}{2}} \cos x dx - \int\limits_{\frac{\pi}{2}}^\pi \cos x dx = 2$.

(iii) $\int\limits_4^9 x\sqrt{x} dx = \int\limits_4^9 x^{\frac{3}{2}} dx = (\frac{2}{5}x^{\frac{5}{2}})|_4^9 = \frac{2}{5}\{(9)^{\frac{5}{2}} - (5)^{\frac{5}{2}}\}$.

(iv) $\int\limits_0^{\frac{\pi}{3}} \sec^2 x dx = \tan x|_0^{\frac{\pi}{3}} = \tan\frac{\pi}{3} - \tan 0 = \sqrt{3}$.

(v) $\int\limits_0^{\frac{\pi}{2}} \sin x dx = -\cos x|_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} + \cos 0 = 1$.

(vi) $\int\limits_{\frac{\pi}{2}}^\pi \cos^2 3x dx = \frac{1}{2}\int\limits_{\frac{\pi}{2}}^\pi (1 + \cos 6x) dx = \frac{1}{2}(x + \frac{1}{6}\sin 6x)|_{\frac{\pi}{2}}^\pi$
$= \frac{1}{2}\{(\pi + 0) - (\frac{\pi}{2} + 0)\} = \frac{\pi}{4}$.

Definite Integrals and Substitution

If $I = \int_a^b f(g(x))g'(x)dx$ and if $F'(x) = f(x)$, then $\frac{d}{dx}(F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$, so $I = F(g(x))|_a^b = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)du$, since $\frac{d}{du}F(u) = f(u)$. In practice, let $u = g(x)$, so $\frac{du}{dx} = g'(x)$ or $du = g'(x)dx$ and then change the limits into u limits.

Examples: (i) $I = \int_0^2 x(x^2+1)^3 dx$; let $u = x^2 + 1$; then $\frac{du}{dx} = 2x$ or $du = 2xdx$.

$x = 0 \Rightarrow u = 1$ and $x = 2 \Rightarrow u = 5$.

Hence $I = \frac{1}{2}\int_1^5 u^3 du = \frac{1}{8}(u^4)|_1^5 = \frac{1}{8}(5^4 - 1^4) = 78$.

(ii) $I = \int_0^{\frac{\pi}{8}} \sin^5 2x \cos 2x dx$; let $u = \sin 2x$; then $\frac{du}{dx} = 2\cos 2x$ or $du = 2\cos 2x dx$.

$x = 0 \Rightarrow u = \sin 0 = 0$ and $x = \frac{\pi}{8} \Rightarrow u = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

Hence $I = \frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}} u^5 du = \frac{1}{12}(u^6)|_0^{\frac{1}{\sqrt{2}}} = \frac{1}{96}$.

(iii) $I = \int_3^4 \frac{2x-6}{(x^2-6x+10)^2}$; let $u = x^2 - 6x + 10$; then $\frac{du}{dx} = 2x - 6$ or $du = (2x-6)dx$. $x = 3 \Rightarrow u = 1$ and $x = 4 \Rightarrow u = 2$.

Hence $I = \int_1^2 \frac{du}{u^2} = \int_1^2 u^{-2}du = \frac{-1}{u}|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$.

Natural Logarithm Function

Consider the function $y = \frac{1}{t}$ for $t > 0$.

We define the natural log (ln) as the function $\ln : \mathbb{R}^+ \longrightarrow \mathbb{R}$, where $\ln x = \int\limits_1^x \frac{1}{t} dt$.

Note:(i) ln is 1-1 and onto.

(ii) $x > 1 \Rightarrow \ln x > 0, \quad 0 < x < 1 \Rightarrow \ln x < 0, \quad \ln 1 = 0.$

(iii) $x \to \infty \Rightarrow \ln x \to \infty, \quad x \to 0 \Rightarrow \ln x \to -\infty.$

(iv) $\frac{d}{dx} \ln x = \frac{1}{x} > 0$ for all $x > 0$ (by the fundamental theorem of calculus) so ln is always increasing.

(v) $\ln(ab) = \ln a + \ln b$.

Proof: $\frac{d}{dx}\ln(ax) = \frac{1}{ax} \times a = \frac{1}{x} = \frac{d}{dx}\ln x \Rightarrow$

$\frac{d}{dx}(\ln(ax) - \ln x) = 0$ so $\ln(ax) - \ln x = c$ (constant);

letting $x = 1$ gives $\ln a - \ln 1 = c$ i.e. $c = \ln a$. Now $\ln(ax) =$

$\ln x + \ln a$, so letting $x = b$ gives $\ln(ab) = \ln a + \ln b$.

(vi) $\ln(\frac{a}{b}) = \ln a - \ln b$.

Proof: $\ln a = \ln(\frac{a}{b} \times b) = \ln(\frac{a}{b}) + \ln b$, so $\ln(\frac{a}{b}) = \ln a - \ln b$.

(vii) $\ln(a^{\frac{m}{n}}) = \frac{m}{n}\ln a$.

Proof: If $m \in \mathbb{N}$, then $\ln(a^m) = \ln a + \ln a + \cdots + \ln a$

(m times) $= m\ln a$.

Now $\ln(a^{-m}) = \ln(\frac{1}{a^m}) = \ln 1 - \ln(a^m) = -m\ln a$.

What about $\ln(a^{\frac{1}{n}})$? Let $b = a^{\frac{1}{n}}$; then $b^n = a$, so $\ln(b^n) = \ln a$

and so $n\ln b = \ln a$ or $\ln b = \frac{1}{n}\ln a$ i.e. $\ln(a^{\frac{1}{n}}) = \frac{1}{n}\ln a$.

Finally, $\ln(a^{\frac{m}{n}}) = \ln((a^{\frac{1}{n}})^m) = m\ln(a^{\frac{1}{n}}) = \frac{m}{n}\ln a$.

Differentiation of Logarithm Functions

If $y = \ln(f(x))$, then $\frac{dy}{dx} = \frac{1}{f(x)}.f'(x)$.

Examples: (i) $y = \ln(\sin x))$;

$\frac{dy}{dx} = \frac{1}{\sin x}.\cos x = \cot x$.

(ii) $y = \ln(x^2.\tan x)$;

$y = \ln(x^2) + \ln(\tan x) = 2\ln x + \ln(\tan x)$ so

$\frac{dy}{dx} = \frac{2}{x} + \frac{1}{\tan x}.\sec^2 x = \frac{2}{x} + \frac{1}{\sin x \cos x}$.

(iii) $y = \ln(\frac{x^2}{x^2+1})$;

$y = \ln(x^2) - \ln(x^2+1) = 2\ln x - \ln(x^2+1)$ so

$\frac{dy}{dx} = \frac{2}{x} - \frac{1}{x^2+1}.2x = \frac{2}{x} - \frac{2x}{x^2+1}$.

(iv) $y = \ln(\frac{x^2.\sin x}{\sqrt{x+1}})$;

$y = \ln(x^2) + \ln(\sin x) - \ln(\sqrt{x+1}) = 2\ln x + \ln(\sin x) - \frac{1}{2}\ln(x+1)$ so $\frac{dy}{dx} = \frac{2}{x} + \frac{1}{\sin x}.\cos x - \frac{1}{2}.\frac{1}{x+1}$

$= \frac{2}{x} + \cot x - \frac{1}{2(x+1)}$.

(v) $y = x\ln x$;

$\frac{dy}{dx} = x.\frac{1}{x} + 1.\ln x = 1 + \ln x$.

Integration of Logarithm Functions

$\frac{d}{dx}\ln x = \frac{1}{x}$ for $x > 0$, so $\int \frac{1}{x}dx = \ln x + c$.

If $x < 0$ what is $\int \frac{1}{x}dx$?

Let $I = \int \frac{1}{x}dx, x < 0$ and let $u = -x$. Then $\frac{du}{dx} = -1$ or $du = -dx$ and $I = \int \frac{-du}{-u} = \int \frac{1}{u}du, u > 0$,

so $I = \ln u + c = \ln(-x) + c$.

In general, $\int \frac{1}{x}dx = \ln \mid x \mid +c$, but usually the domain of integration is assumed to be positive and we omit the mod signs. Any integral of the form $I = \int \frac{f'(x)}{f(x)}dx$ is done by letting $u = f(x)$ etc.

Examples: (i) $I = \int \frac{1}{3x+2}dx$; let $u = 3x + 2$; then $\frac{du}{dx} = 3$ or $du = 3dx$. Hence $I = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln u + c = \frac{1}{3}\ln(3x + 2) + c$.

(ii) $I = \int \frac{3x^2}{x^3+1}dx$; let $u = x^3 + 1$; then $\frac{du}{dx} = 3x^2$

or $du = 3x^2dx$. Hence $I = \int \frac{du}{u} = \ln u + c = \ln(x^3 + 1) + c$.

(iii) $I = \int\limits_1^2 \frac{2x}{x^2+1}dx$; let $u = x^2 + 1$; then $\frac{du}{dx} = 2x$

or $du = 2xdx$.

$x = 1 \Rightarrow u = 2$ and $x = 2 \Rightarrow u = 5$.

Hence $I = \int\limits_2^5 \frac{du}{u} = (\ln u)|_2^5 = \ln 5 - \ln 2 = \ln(\frac{5}{2})$.

(iv) $I = \int\limits_5^7 \frac{x}{x+5} dx$; let $u = x+5$; then $\frac{du}{dx} = 1$ or $du = dx$.
Also $x = u - 5$.

$x = 5 \Rightarrow u = 10$ and $x = 7 \Rightarrow u = 12$.

Hence $I = \int\limits_{10}^{12} \frac{u-5}{u} du = \int\limits_{10}^{12} du - \int\limits_{10}^{12} \frac{5}{u} du = (u)|_{10}^{12} - 5(\ln u)|_{10}^{12} = 2 - \ln(\frac{6}{5})$.

(v) $I = \int\limits_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan x dx = \int\limits_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{\cos x} dx$; let $u = \cos x$;
then $\frac{du}{dx} = -\sin x$ or $du = -\sin x dx$.

$x = \frac{\pi}{6} \Rightarrow u = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $x = \frac{\pi}{3} \Rightarrow u = \cos\frac{\pi}{3} = \frac{1}{2}$.

Hence $I = -\int\limits_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{du}{u} = \int\limits_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{du}{u} = (\ln u)\Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \ln(\frac{\sqrt{3}}{2}) - \ln(\frac{1}{2})$

$= \ln(\frac{(\frac{\sqrt{3}}{2}}{\frac{1}{2}}) = \ln(\sqrt{3}) = \frac{1}{2}\ln 3$.

Exponential and Power Functions

$\ln : \mathbb{R}^+ \longrightarrow \mathbb{R} : x \mapsto \int\limits_1^x \frac{1}{t} dt$ is 1-1 and onto, so its inverse exists. Call it exponential, exp, i.e. $\exp : \mathbb{R} \longrightarrow \mathbb{R}^+$ is $\ln^{-1}$.

We have $y = \exp x \Leftrightarrow \ln y = x$.

Recall the chain rule for derivatives: $(fog)'(x) = f'(g(x)).g'(x)$.

Since $(f^{-1}of)(x) = x$, we get $(f^{-1}of)'(x) = 1$,

so $(f^{-1})'(f(x)).f'(x) = 1$ or $(f^{-1})'(f(x)) = \frac{1}{f'(x)}$.

If $y = f(x)$ then $x = f^{-1}(y)$ and, in the other notation, we have $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$.

Now $y = \exp x \Leftrightarrow \ln y = x$ so $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{1}{y}} = y = \exp x$ i.e. $\frac{d}{dx} \exp x = \exp x$. We can also see this by using implicit differentiation since $\ln y = x \Rightarrow \frac{1}{y}\frac{dy}{dx} = 1$ and so $\frac{dy}{dx} = y = \exp x$.

Note: (i) exp is 1-1 and onto.

(ii) $\exp 0 = 1$ since $\ln 1 = 0$.

(iii) $x \to \infty \Rightarrow \exp x \to \infty, \quad x \to -\infty \Rightarrow \exp x \to 0$.

(iv) $\exp(a+b) = \exp a . \exp b$.

Proof: Let $x = \exp a$ and let $y = \exp b$. Then $\ln x = a$ and $\ln y = b$, so that $\ln(xy) = \ln x + \ln y = a + b$ and then $xy = \exp(a+b)$ i.e. $\exp(a+b) = \exp a . \exp b$.

(v) $\exp(a-b) = \frac{\exp a}{\exp b}$.

Proof: Let $x = \exp a$ and let $y = \exp b$. Then $\ln x = a$ and $\ln y = b$, so that $\ln(\frac{x}{y}) = \ln x - \ln y = a - b$ and then $\frac{x}{y} = \exp(a-b)$ i.e. $\exp(a-b) = \frac{\exp a}{\exp b}$.

It seems that exp is behaving like a "power function", at least if a and b above are rational numbers. We now define what we mean by any real number power, not just rational.

Let $a > 0$ and $p, q \in \mathbb{Z}$. We know what $a^{\frac{p}{q}}$ means, namely $a^{\frac{p}{q}} = (a^p)^{\frac{1}{q}}$. Also, of course, $a^{\frac{p}{q}} = \exp(\ln(a^{\frac{p}{q}})) = \exp(\frac{p}{q} \ln a)$. This motivates the following

Definition: For any $x \in \mathbb{R}$, $\quad a^x = \exp(x \ln a)$.

Note that $\ln a$ is defined since $a > 0$ and then $\exp(x \ln a)$ is defined for all $x \in \mathbb{R}$, so this is a valid definition. Also it agrees with the definition of a^x when x is rational. We have extended the definition of powers from $\mathbb{Q}$ to all of $\mathbb{R}$.

Now for any $x \in \mathbb{R}$ we have $\ln(a^x) = \ln(\exp(x \ln a)) = x \ln a$.

Definition: $e = \exp 1$.

Now $e^x = \exp(x \ln e) = \exp(x.1) = \exp x$, so exp itself is a power function, namely $\exp x = e^x$. Also $a^x = e^{x \ln a}$.

Definition: We define $\log_a$, log to the base a, as the inverse of a^x i.e. $y = \log_a x \Leftrightarrow a^y = x$.

Since ln is the inverse of $\exp x = e^x$, we see that ln is, in fact, $\log_e$, called log to the natural base, i.e. $\ln x = \log_e x$.

We can easily change from ony base a to the natural base:

$y = \log_a x \Rightarrow a^y = x \Rightarrow \ln(a^y) = \ln x \Rightarrow y \ln a = \ln x$

$\Rightarrow y = \frac{\ln x}{\ln a}$ i.e. $\log_a x = \frac{\ln x}{\ln a}$.

We have the following diagram for powers and logs:

Differentiation of Exponential Functions

$y = e^{f(x)} \Rightarrow \frac{dy}{dx} = e^{f(x)}.f'(x)$.

Examples: (i) $y = e^{4x^2}$; $\quad \frac{dy}{dx} = e^{4x^2}.8x$.

(ii) $y = x^2.e^{\cos x}$; $\quad \frac{dy}{dx} = x^2.e^{\cos x}.(-\sin x) + 2x.e^{\cos x}$

$= x.e^{\cos x}(2 - x\sin x)$.

(iii) $y = \ln(\frac{e^x}{1+e^x}) = \ln(e^x) - \ln(1+e^x) = x - \ln(1+e^x)$;

$\frac{dy}{dx} = 1 - \frac{1}{1+e^x}.e^x = \frac{1}{1+e^x}$.

(iv) $y = \ln(e^x.\sin x) = \ln(e^x) + \ln(\sin x) = x + \ln(\sin x)$;

$\frac{dy}{dx} = 1 + \frac{1}{\sin x}.\cos x = 1 + \cot x$.

(v) $y = a^x = e^{x\ln a}$; $\quad \frac{dy}{dx} = e^{x\ln a}.\ln a = \ln a.a^x$.

We could also use implicit differentiation: $\quad y = a^x \Rightarrow \ln y = \ln(a^x) = x\ln a$, so $\frac{1}{y}.\frac{dy}{dx} = \ln a$ and hence $\frac{dy}{dx} = \ln a.y = \ln a.a^x$.

(vi) $y = \log_a x = \frac{\ln x}{\ln a}$; $\quad \frac{dy}{dx} = \frac{1}{\ln a}.\frac{1}{x}$.

Integration of Exponential Functions

$\frac{d}{dx}e^x = e^x \Rightarrow \int e^x dx = e^x + c.$

Any integral of the form $I = \int e^{f(x)}.f'(x)dx$ is done by letting $u = f(x)$ etc.

Examples: (i) $I = \int e^{3x-2}dx;$ let $u = 3x - 2$; then $\frac{du}{dx} = 3$ or $du = 3dx$. Hence $I = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + c = \frac{1}{3}e^{3x-2} + c.$

(ii) $I = \int_0^1 4xe^{x^2}dx;$ let $u = x^2$; then $\frac{du}{dx} = 2x$ or $du = 2xdx$.

$x = 0 \Rightarrow u = 0$ and $x = 1 \Rightarrow u = 1$.

Hence $I = 2\int_0^1 e^u du = 2(e^u)|_0^1 = 2(e - 1)$.

(iii) $I = \int_0^{\frac{\pi}{2}} e^{\sin x}\cos x dx;$ let $u = \sin x$; then $\frac{du}{dx} = \cos x$ or $du = \cos x dx$.

$x = 0 \Rightarrow u = 0$ and $x = \frac{\pi}{2} \Rightarrow u = 1$.

Hence $I = \int_0^1 e^u du = (e^u)|_0^1 = (e - 1)$.

(iv) $I = \int_0^1 \frac{e^{2x}+4}{e^x}dx = \int_0^1 (e^x + 4e^{-x})dx = (e^x - 4e^{-x})|_0^1 = \{(e - \frac{4}{e}) - (1 - 4)\} = e - \frac{4}{e} + 3.$

(v) $I = \int_0^1 \frac{e^x}{e^x+1}dx;$ let $u = e^x + 1$; then $\frac{du}{dx} = e^x$ or $du = e^x dx$.

$x = 0 \Rightarrow u = 2$ and $x = 1 \Rightarrow u = e + 1$.

Hence $I = \int_2^{e+1} \frac{1}{u}du = (\ln u)|_2^{e+1} = \ln(\frac{e+1}{2})$.

Differential Equations

A linear first order differential equation is an equation of the form

$$f(x)\frac{dy}{dx} + g(x)y = h(x)$$

where $f(x) \neq 0$ on some interval I. A solution of the equation is any function $y(x)$ that satisfies it. The equation is homogeneous if $h(x) = 0$. We shall consider only homogeneous equations here. The equation is called linear since if $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation, then so is $y_1(x) + y_2(x)$ and $y_1(x) - y_2(x)$ and $\alpha y_1(x)$ for all $\alpha \in \mathbb{R}$. We solve the equation

$$f(x)\frac{dy}{dx} + g(x)y = 0$$

by "separating the variables":

$f(x)\frac{dy}{dx} = -g(x)y$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -\frac{g(x)}{f(x)}$, assuming $y \neq 0$ on I,

$\Rightarrow \int \frac{1}{y}\frac{dy}{dx} dx = -\int \frac{g(x)}{f(x)} dx$

$\Rightarrow \int \frac{1}{y} dy = -\int \frac{g(x)}{f(x)} dx$, by the substitution formula,

$\Rightarrow \ln y = -\int \frac{g(x)}{f(x)} dx = -j(x) + c$, say, assuming $y > 0$ on I,

$\Rightarrow y = e^{-j(x)+c} = e^{-j(x)}e^c = De^{-j(x)}$, letting $D = e^c$.

Examples: (i) $x\frac{dy}{dx} + y = 0, \quad I = (0, \infty)$.

$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{1}{y}\frac{dy}{dx} = -\frac{1}{x}$

$\Rightarrow \int \frac{1}{y}\frac{dy}{dx}dx = \int \frac{1}{y}dy = -\int \frac{1}{x}dx \Rightarrow \ln y = -\ln x + c$

$\Rightarrow y = e^{-lnx+c} = De^{-lnx} = De^{\ln\frac{1}{x}} = D.\frac{1}{x}, \quad D$ a constant.

(ii) $(3x^2+1)\frac{dy}{dx} - 2xy = 0, \quad I = (-\infty, \infty)$.

$(3x^2+1)\frac{dy}{dx} - 2xy = 0 \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{2x}{3x^2+1}$

$\Rightarrow \int \frac{1}{y}\frac{dy}{dx}dx = \int \frac{1}{y}dy = \int \frac{2x}{3x^2+1}dx \Rightarrow \ln y = \frac{1}{3}\ln(3x^2+1) + c$

$\Rightarrow y = De^{\frac{1}{3}\ln(3x^2+1)} = De^{\ln(3x^2+1)^{\frac{1}{3}}} = D(3x^2+1)^{\frac{1}{3}}$.

Exercise: (i) $x\frac{dy}{dx} + 2y = 0, \quad I = (0, \infty)$

(ii) $\frac{dy}{dx} - (\tan x)y = 0, \quad I = (-\infty, \infty)$.

Every linear first order differential equation has infinitely many solutions (take different values for the constant D). However, if we specify that y must be a particular value for some given value of x, $y(x_0) = y_0$ say, then we get a unique solution since we can solve for D. The equation $f(x)\frac{dy}{dx} + g(x)y = 0$ with the condition $y(x_0) = y_0$ is called an initial-value problem.

Examples: (i) $2\frac{dy}{dx}+3y = 0, \quad y(-3) = -3, \quad I = (-\infty, \infty)$.

$2\frac{dy}{dx} + 3y = 0 \Rightarrow \int \frac{1}{y}dy = -\int \frac{3}{2}dx$

$\Rightarrow \ln y = -\frac{3}{2}x + c \Rightarrow y = De^{-\frac{3}{2}x}$;

Now $y(-3) = -3 \Rightarrow -3 = De^{\frac{3}{2}.3} = De^{\frac{9}{2}}$, so $D = -3e^{-\frac{9}{2}}$, and hence $y = -3e^{-\frac{9}{2}}e^{-\frac{3}{2}x} = -3e^{-\frac{3}{2}(x+3)}$.

(ii) $\frac{dy}{dx} + (\cot x)y = 0, \quad y(\frac{\pi}{2}) = 1, \quad I = (0, \pi)$.

$\frac{dy}{dx} + (\cot x)y = 0 \Rightarrow \int \frac{1}{y}dy = -\int \cot x dx$

$\Rightarrow \ln y = -\ln(\sin x) + c \Rightarrow y = De^{-\ln(\sin x)} = De^{\ln(\frac{1}{\sin x})} = D.\frac{1}{\sin x}$;

Now $y(\frac{\pi}{2}) = 1 \Rightarrow 1 = D(\frac{1}{1})$, so $D = 1$ and $y = \frac{1}{\sin x}$.

Definition: A quantity is said to have an exponential growth (decay) model if it increases (decreases) at a rate that is proportional to the amount of the quantity present at any given time. Mathematically we have

$\frac{dy}{dt} = ky, \quad k > 0$ (growth) and $\frac{dy}{dt} = -ky, \quad k > 0$ (decay).

k is called the growth (decay) constant.

$\frac{dy}{dt} = ky \Rightarrow \int \frac{dy}{y} = k \int dt \Rightarrow \ln y = kt + c$

$\Rightarrow y = e^{kt}e^{c} = De^{kt}$, for some constant D. Taking $t = 0$ gives $y(0) = De^{0}$, so $D = y(0)$, the initial value of the quantity.

Similarly, if $\frac{dy}{dt} = -ky, \quad k > 0$, we get $y = De^{-kt}$.

k is often called the relative growth rate, $\frac{\frac{dy}{dt}}{y}$, i.e. the growth rate as a fraction of the quantity, which is constant over time.

It is usually given as a percentage e.g. a relative growth rate of 5 percent per unit of time means that $k = .05$.

The time required for the initial quantity to double is called the doubling time : Suppose $y = y_0 e^{kt}$, where $y_0 = y(0)$ is the initial size. Then $2y_0 = y_0 e^{kt} \Rightarrow 2 = e^{kt}$, so that $kt = \ln 2$ and $t = \frac{\ln 2}{k}$ is the doubling time. This is usually denoted by T i.e. $T = \frac{\ln 2}{k}$.

For decaying the time taken for half the quantity to decay is called the half-life: we get $\frac{1}{2} y_0 = y_0 e^{-kt} \Rightarrow \frac{1}{2} = e^{-kt}$, so that $-kt = -\ln 2$ and $t = \frac{\ln 2}{k}$ again.

Examples: (i) Suppose that an initial population of a colony of bacteria is 10,000 and that the colony grows exponentially at the rate of 1 per cent per hour and that $y = y(t)$ is the number of bacteria present t hours later.

(a) Find an initial-value problem whose solution is $y(t)$.

(b) Solve for $y(t)$.

(c) How long does it take the population to double?

(d) How long does it take the population to reach45,000?

(a) $\frac{dy}{dt} = ky, \quad k > 0, \quad y(0) = 10,000.$

(b) $\frac{dy}{dt} = ky \Rightarrow \int \frac{dy}{y} = k \int dt \Rightarrow \ln y = kt + c$

$\Rightarrow y = e^{kt}e^{c} = De^{kt}$, for some constant D.

Now $k = .01 = \frac{1}{100}$, so $y = De^{\frac{1}{100}t}$; to find D we have

$y(0) = 10,000 \Rightarrow 10,000 = De^0 = D$. The full solution is

$y(t) = 10,000e^{\frac{1}{100}t}$.

(c) $20,000 = 10,000e^{\frac{1}{100}t} \Rightarrow e^{\frac{1}{100}t} = 2$, so $T = 100\ln 2$ hours.

We could just use the formula $T = \frac{\ln 2}{k}$.

(d) $45,000 = 10,000e^{\frac{1}{100}t} \Rightarrow e^{\frac{1}{100}t} = 4.5$, so $t = 100\ln 4.5$ hours.

(ii) A cell of e.coli divides into two cells every 20 minutes when placed in a nutrient culture. Let $y = y(t)$ be the number of cells present t minutes after a single cell is placed in the culture and assume that the growth rate is approximated by a continuous exponential growth model.

(a) Find an initial-value problem whose solution is $y(t)$.

(b) Solve for $y(t)$.

(c) How many cells are present after 2 hours?

(d) How long does it take for the number of cells to reach

1,000,000?

(a) $\frac{dy}{dt} = ky, \quad k > 0, \quad y(0) = 1.$

(b) $\frac{dy}{dt} = ky \Rightarrow \int \frac{dy}{y} = k \int dt \Rightarrow \ln y = kt + c$

$\Rightarrow y = e^{kt} e^c = De^{kt}$, for some constant D. Also $y(0) = 1 \Rightarrow$ $D = 1$, so $y(t) = e^{kt}$.

Now $T = 20$, so $20 = \frac{\ln 2}{k}$ or $k = \frac{\ln 2}{20}$ and so $y(t) = e^{\frac{\ln 2}{20} t}$.

(c) $y(120) = e^{\frac{\ln 2}{20} \cdot 120} = e^{6 \ln 2}$ cells.

(d) $1,000,000 = e^{\frac{\ln 2}{20} t} \Rightarrow \frac{\ln 2}{20} t = \ln(10^6) = 6 \ln 10,$

so $t = \frac{120 \ln 10}{\ln 2}$ minutes.

(iii) In a certain culture of bacteria the number of bacteria increased sixfold in 10 hours. How long did it take for the population to double, assuming that the growth rate is approximated by a continuous exponential growth model?

$y = y_0 e^{kt} \Rightarrow 6y_0 = y_0 e^{10k}$, so $k = \frac{\ln 6}{10}$. Doubling time is $T = \frac{\ln 2}{k} = \frac{10 \ln 2}{\ln 6}$ hours. Alternatively, $2y_0 = y_0 e^{kT} \Rightarrow e^{kT} = 2$, so $T = \frac{\ln 2}{k} = \frac{10 \ln 2}{\ln 6}$ hours.