MAU33E01

Lecture Notes

on

Fourier Analysis

and

Partial Differential Equations

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Main Text: Kreyszig; Advanced Engineering Mathematics

Other Texts: Nagle and Saff, Zill and Wright, Farlow,

Brown and Churchill, Stroud

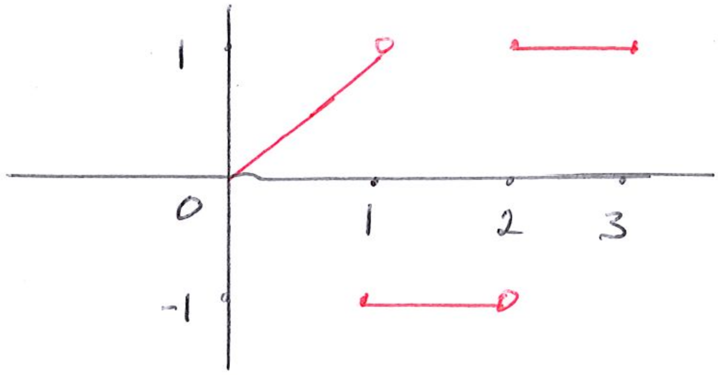
Online Notes: Ruth Baker (Oxford), Paul's online notes

FOURIER SERIES

Some Definitions

Definition: A function $f : [a, b] \to \mathbb{R}$ is piecewise continuous if $[a, b]$ can be divided into a finite number of subintervals on the interiors of which f is continuous and has finite left-hand and right-hand limits at all the endpoints of these subintervals.

Example:



f is piecewise continuous on $[0, 3]$.

Let $C_p[a, b]$ denote the set of all piecewise continuouos functions on $[a, b]$. Then $C_p[a, b]$ is a vector space under function addition and scalar multiplication. If x_0 is an endpoint of some subinterval we write $\lim_{x \to x_0^+} f(x) = f(x_0+)$ and $\lim_{x \to x_0^-} f(x) = f(x_0-)$. If f is continuous at x_0, then both of those limits are the same.

Definition: $f : \mathbb{R} \to \mathbb{R}$ is periodic if there exists some $T \neq 0$ such that $f(x + T) = f(x)$ for all $x \in \mathbb{R}$. The smallest positive such T is called the period of f.

Example: (i) $f(x) = \sin x$ is periodic with period 2π, so $f(x) = \sin nx$ is periodic with period $\frac{2\pi}{n}$ since $\sin n(x + \frac{2\pi}{n}) = \sin(nx + 2\pi) = \sin nx$.

(ii) $f(x) = c$, a constant, is periodic but does not have a period.

(iii) $f(x) = x^2$ is not periodic.

Note: The sum and the product of periodic functions is usually periodic.

Definition: $f : \mathbb{R} \to \mathbb{R}$ is even if $f(-x) = f(x)$ for all x and f is odd if $f(-x) = -f(x)$ for all x.

Example: cos is even and sin is odd.

Note that (even)(even) = even, (odd)(odd) = even and (even)(odd) = odd.

Theorem: If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$ and if f is odd, then $\int_{-a}^{a} f(x)dx = 0$.

Proof: $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$. Making the substitution $u = -x$ in the first integral gives $\int_{0}^{a} f(-u)du =$

$\int_0^a f(-x)dx$. Hence both results.

We will need to consider the vector space nature of $C_p[a,b]$, so we recall the abstract definition of a vector space.

Definition: A set V along with an operation $+ : V \times V \to V$ (addition) and an operation $. : \mathbb{R} \times V \to V$ (scalar multiplication) is called a real vector space if the following properties hold

(i) $u + v = v + u \; \forall u, v \in V$ (commutivity)

(ii) $(u + v) + w = u + (v + w) \; \forall u, v, w \in V$ (associativity)

(iii) $\exists$ an element $0 \in V$ such that $u + 0 = 0 + u \; \forall u \in V$ (zero element)

(iv) Given any $v \in V \; \exists - v \in V$ such that $v + (-v) = (-v) + v = 0$ (inverse element)

(v) $\alpha.(u + v) = \alpha.u + \alpha.v \; \forall \alpha \in \mathbb{R}, u, v \in V$

(vi) $(\alpha + \beta).u = \alpha.u + \beta.u \; \forall \alpha, \beta \in \mathbb{R}, u \in V$

(vii) $\alpha.(\beta.u) = (\alpha\beta).u \; \forall \alpha, \beta \in \mathbb{R}, u \in V$ ((v), (vi), (vii) imply distributivity)

(viii) $1.u = u \; \forall u \in V$.

$\mathbb{R}^3$

Recall the structure of $\mathbb{R}^3$:

$\mathbb{R}^3 = \{(a,b,c)|a,b,c \in \mathbb{R}\}$, where we write $\vec{v} = (a,b,c)$. If $\vec{i} = (1,0,0), \vec{j} = (0,1,0)$ and $\vec{k} = (0,0,1)$, then $\vec{v} = (a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1) = a\vec{i} + b\vec{j} + c\vec{k}$, where addition and scalar mutiplication is defined componentwise. We say that $\mathbb{R}^3$ is a vector space with a basis $\{\vec{i}, \vec{j}, \vec{k}\}$.

Thr scalar (dot) product on $\mathbb{R}^3$ is defined by $(a,b,c).(d,e,f) = ad + be + cf$ so that $\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1$.

We define $\vec{u} \perp \vec{v}$ if and only if $\vec{u}.\vec{v} = 0$, so $\vec{i}, \vec{j}$ and $\vec{k}$ are mutually perpendicular.

The norm or length of $\vec{v}$ is defined by $\|\vec{v}\| = \sqrt{\vec{v}.\vec{v}} = \sqrt{a^2 + b^2 + c^2}$ and the distance from $\vec{u}$ to $\vec{v}$ is $\|\vec{u} - \vec{v}\|$.

Note that if $\vec{v} = (a,b,c) = a\vec{i} + b\vec{j} + c\vec{k}$, then $a = \vec{v}.\vec{i}, b = \vec{v}.\vec{j}$ and $c = \vec{v}.\vec{k}$.

Since every basis of $\mathbb{R}^3$ has three elements we say that $\mathbb{R}^3$ is three-dimensional. Other vector spaces are infinite-dimensional e.g. $C_p[a,b]$.

The Vector Space $C_p[a, b]$

We define an inner product on $C_p[a, b]$ by $(f, g) = \int_a^b f(x)g(x)dx$. Then $f \perp g$ if $(f, g) = 0$. In this case we say that f is orthogonal to g. We write "the norm of f" to be $\|f\| = \sqrt{(f, f)} = (\int_a^b f^2(x)dx)^{\frac{1}{2}}$ and "the distance from f to g" as $\|f - g\| = (\int_a^b (f(x) - g(x))^2 dx)^{\frac{1}{2}}$. We say that f is normal if $\|f\| = 1$ i.e. $\int_a^b f^2(x)dx = 1$.

A set of functions $\{\phi_0, \phi_1, ..., \phi_n, ...\}$ is orthonormal if $\|\phi_n\| = 1$ for all n and $(\phi_n, \phi_m) = 0$ for all $n \neq m$.

Given a set of orthogonal functions $\{\psi_0, \psi_1, ..., \psi_n, ...\}$ we get an orthonormal set by taking $\phi_n = \frac{\psi_n}{\|\psi_n\|}$ for all n.

Example: Consider $[0, \pi]$ and $\{1, \cos x, \cos 2x, ..., \cos nx, ...\}$. $(\int_0^\pi 1^2 dx)^{\frac{1}{2}} = \sqrt{\pi}$. $(\int_0^\pi \cos^2 nx dx)^{\frac{1}{2}} = (\frac{1}{2}\int_0^\pi (1+\cos 2nx)dx)^{\frac{1}{2}} = (\frac{1}{2}(x + \frac{1}{2n}\sin 2nx)|_0^\pi)^{\frac{1}{2}} = (\frac{1}{2}[(\pi - 0) - (0 - 0)])^{\frac{1}{2}} = \sqrt{\frac{\pi}{2}}$. Also we get $\int_0^\pi \cos nx \cos mx dx = 0$ for all $n \neq m$. Hence $\{1, \cos x, \cos 2x, ..., \cos nx, ...\}$ is an orthogonal set and $\{\frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}} \cos nx | n = 1, 2, 3...\}$ is an orthonormal set in $C_p[0, \pi]$.

Similarly the set $\{\sin x, \sin 2x, ..., \sin nx, ...\}$ is an orthogonal set and $\{\sqrt{\frac{2}{\pi}}\sin nx | n = 1, 2, 3...\}$ is an orthonormal set in $C_p[0, \pi]$.

Generalized Fourier Series

Let $\{\phi_n\}$ be an orthonormal set in $C_p[a, b]$. Consider any $f \in C_p[a, b]$ and let $c_n = (f, \phi_n)$ for all n. Is it possible to write f as an infinite series $f = \sum_{n=0}^{\infty} c_n \phi_n$, where convergence is in the norm $\| \quad \|$, or better still, $f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$, for all $x \in [a, b]$, where convergence is now the usual convergence in $\mathbb{R}$? This is analagous to any vector $\vec{v} \in \mathbb{R}^3$ being written as $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$, where $a = \vec{v}.\vec{i}$, $b = \vec{v}.\vec{j}$ and $c = \vec{v}.\vec{k}$.

We call the series $\sum_{n=0}^{\infty} c_n \phi_n(x)$, where $c_n = (f, \phi_n) = \int_a^b f(x)\phi_n(x)dx$ for all n, the generalized Fourier series of $f(x)$. We don't yet know whether this series actually converges for any $x \in [a, b]$ or, if it does converge, whether the limit is equal to $f(x)$. We write $f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x)$ to mean that the right-hand side is the series so obtained from $f(x)$ leaving aside the questions of convergence for the moment. The theory of Fourier series tells us that for some subsets of $C_p[a, b]$ and for some orthonormal sets $\{\phi_n(x)\}$ we do in fact have convergence and equality.

Consider $C_p[-\pi, \pi]$ and the set $\{\cos nx, \sin mx\}, \quad n, m \in \mathbb{N}$.

We have $\int\limits_{-\pi}^{\pi} \sin mx \cos nx dx = 0$ for all m, n; $\int\limits_{-\pi}^{\pi} \sin mx \sin nx dx$ and $\int\limits_{-\pi}^{\pi} \cos mx \cos nx dx$ both $= \pi$, if $m = n$ and 0 if $m \neq n$.

Also for $n = 0$ we get $\int\limits_{-\pi}^{\pi} \cos^2 0x dx = 2\pi$.

Hence the set $\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx\}, n = 1, 2, 3,$ forms an orthonormal set in $C_p[-\pi, \pi]$.

Letting $c_0 = \frac{1}{\sqrt{2\pi}} \int\limits_{-\pi}^{\pi} f(x)dx$, $c_n = \frac{1}{\sqrt{\pi}} \int\limits_{-\pi}^{\pi} f(x) \cos nx dx$ and $d_n = \frac{1}{\sqrt{\pi}} \int\limits_{-\pi}^{\pi} f(x) \sin nx dx$ we have

$$f(x) \sim \frac{c_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} (c_n \frac{\cos nx}{\sqrt{\pi}} + d_n \frac{\sin nx}{\sqrt{\pi}}),$$

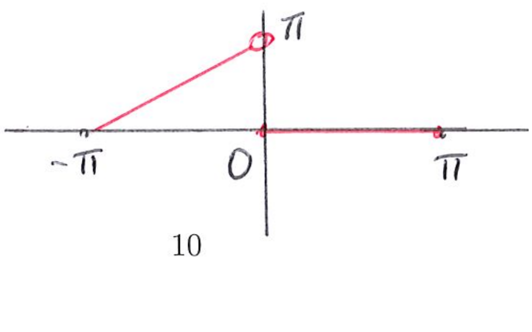
so that $f(x) \sim \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, where

$\frac{a_0}{2} = \frac{c_0}{\sqrt{2\pi}}$, so $a_0 = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(x)dx$, $\quad a_n = \frac{c_n}{\sqrt{\pi}} = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(x) \cos nx dx$

and $b_n = \frac{d_n}{\sqrt{\pi}} = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(x) \sin nx dx$. This is called the Fourier series associated with $f(x)$.

Example:

$$f(x) = \begin{cases} x + \pi, & -\pi \leq x < 0 \\ 0, & 0 \leq x \leq \pi \end{cases}$$



$a_0 = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int\limits_{-\pi}^{0}(x+\pi)dx = \frac{1}{\pi}(\frac{1}{2}x^2+\pi x)|_{-\pi}^{0} =$
$-\frac{1}{\pi}(\frac{1}{2}\pi^2-\pi^2) = \frac{\pi}{2}.$
$a_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int\limits_{-\pi}^{0}(x+\pi)\cos nx dx =$

$\frac{1}{\pi}\int\limits_{-\pi}^{0} x\cos nx dx + \int\limits_{-\pi}^{0}\cos nx dx = \frac{1}{\pi}[(\frac{1}{n}x\sin nx)|_{-\pi}^{0} - \frac{1}{n}\int\limits_{-\pi}^{0}\sin nx dx]$

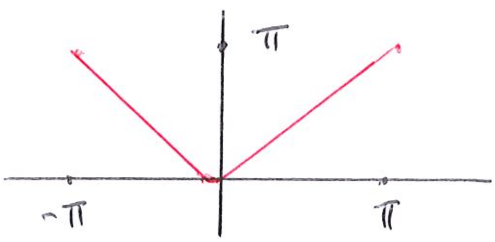
$+ (\frac{1}{n}\sin nx)|_{-\pi}^{0} = -\frac{1}{\pi n}\int\limits_{-\pi}^{0}\sin nx dx = -\frac{1}{\pi n}(-\frac{1}{n}\cos nx)|_{-\pi}^{0} =$

$\frac{1}{\pi n^2}(1-\cos n\pi).$
$b_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int\limits_{-\pi}^{0}(x+\pi)\sin nx dx =$

$\frac{1}{\pi}\int\limits_{-\pi}^{0} x\sin nx dx + \int\limits_{-\pi}^{0}\sin nx dx = \frac{1}{\pi}[(-\frac{1}{n}x\cos nx)|_{-\pi}^{0} + \frac{1}{n}\int\limits_{-\pi}^{0}\cos nx dx]$
$-(\frac{1}{n}\cos nx)|_{-\pi}^{0} = \frac{1}{\pi}[(-\frac{\pi}{n}\cos n\pi) + (\frac{1}{n^2}\sin nx)|_{-\pi}^{0}]$
$+(-\frac{1}{n}+\frac{1}{n}\cos n\pi) = -\frac{1}{n}.$
Therefore $f(x) \sim \frac{\pi}{4} + \sum\limits_{n=1}^{\infty}[\frac{1}{\pi n^2}(1-\cos n\pi)\cos nx - \frac{1}{n}\sin nx].$

Note: Recall that if $f(x)$ is an odd function and c is any real number, then $\int\limits_{-c}^{c} f(x)dx = 0$. Similarly, if $f(x)$ is an even function, we get that $\int\limits_{-c}^{c} f(x)dx = 2\int\limits_{0}^{c} f(x)dx$.

Example: $f(x) = |x|, \quad x \in [-\pi, \pi].$



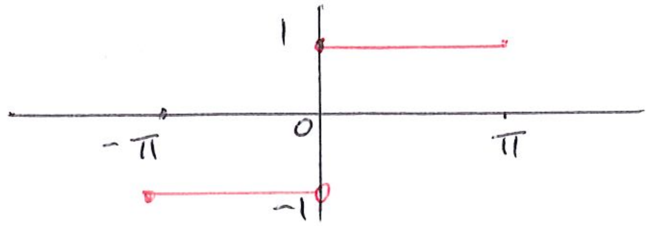
Since $|x|$ is even we get $a_0 = \frac{2}{\pi}\int_0^\pi x dx = \pi$. Also $a_n = \frac{2}{\pi}\int_0^\pi x \cos nx dx = \frac{2}{\pi n^2}(\cos n\pi - 1)$ and $b_n = 0$ for all $n \geq 1$.

Therefore $|x| \sim \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2}(\cos n\pi - 1)\cos nx$.

In general, if $f(x)$ is even, then $b_n = 0$ for all $n \geq 1$ and $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx$ for all $n \geq 0$. Hence $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$. If $f(x)$ is odd, then $a_n = 0$ for all $n \geq 0$ and $b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx dx$ for all $n \geq 1$. Hence $f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$.

Example:

$$f(x) = \begin{cases} -1, & -\pi \leq x < 0 \\ 1, & 0 \leq x \leq \pi \end{cases}$$



Note that $f(x)$ is an odd function on $[-\pi, \pi]$. Hence $a_n = 0$ for all $n \geq 0$. Also $b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx dx$ for all $n \geq 1$, so $b_n = \frac{2}{\pi}\int_0^\pi 1.\sin nx dx = \frac{2}{\pi}(-\frac{1}{n}\cos nx)|_0^\pi = \frac{2}{\pi n}(1 - \cos n\pi)$.

Therefore $f(x) \sim \sum_{n=1}^{\infty} \frac{2}{\pi n}(1 - \cos n\pi)\sin nx$.

Convergence of Fourier Series

We now discuss the convergence of Fourier series. There are two types of convergence

(i) Convergence in the norm or convergence in the mean and

(ii) Pointwise convergence.

(i) Suppose the $f \in C_p[-\pi, \pi]$; Then we write

$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, as before.

Let $S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$. Convergence in the norm (mean) means that $S_N \to f$ as $N \to \infty$ in $\| \quad \|$ i.e. $\|f - S_N\| \to 0$ as $N \to \infty$ or $\int_{-\pi}^{\pi} (f(x) - S_N(x))^2 dx \to 0$ as $N \to \infty$. We have

Theorem: If $f \in C_p[-\pi, \pi]$ (in fact f can belong to a larger space called square-integrable functions), then $S_N \to f$ in the norm.

This result is theoretically interesting but not very useful for our purposes.

(ii) Now suppose that both f and $f' \in C_p[-\pi, \pi]$. Such a function is called piecewise smooth. We have

Theorem: If f is piecewise smooth on $[-\pi, \pi]$, then

$S_N(x) \to \frac{1}{2}(f(x+) + f(x-))$ or $\frac{1}{2}(f(x+) + f(x-)) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ for all $x \in (-\pi, \pi)$. If f is continuous at x, then $\frac{1}{2}(f(x+) + f(x-)) = f(x)$, so that $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$.
If $f : \mathbb{R} \to \mathbb{R}$ is piecewise smooth and periodic of period 2π, then this result applies for all $x \in \mathbb{R}$.

Example:

$$f(x) = \begin{cases} -1, & -\pi \leq x < 0 \\ 1, & 0 \leq x \leq \pi \end{cases}$$

For all $x \in (-\pi, \pi)$ we have $\frac{1}{2}(f(x+) + f(x-)) = \sum_{n=1}^{\infty} \frac{2}{\pi n}(1 - \cos n\pi) \sin nx$. So $x = 0$ gives $\frac{1-1}{2} = 0$ and at $x = \frac{\pi}{2}$ we get $1 = \sum_{n=1}^{\infty} \frac{2}{\pi n}(1 - \cos n\pi) \sin n\frac{\pi}{2}$
$= \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin(2m+1)\frac{\pi}{2} = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}$, so $\frac{\pi}{4} = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}$.

Functions of any Period

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is piecewise smooth and periodic of period $2L$. Let $u = \frac{\pi}{L}x$ and set $g(u) = f(x) = f(\frac{L}{\pi}u)$. We have $g : [-\pi, \pi] \to [-L, L] \to \mathbb{R}$, where $g = f \circ \frac{L}{\pi}$ on $\mathbb{R}$. Then $g(u + 2\pi) = f(\frac{L}{\pi}(u + 2\pi)) = f(\frac{L}{\pi}u + 2L) = f(\frac{L}{\pi}u) = g(u)$, so g has period 2π.

Therefore $g(u) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty}(a_n \cos nu + b_n \sin nu)$, with $a_0 = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} g(u)du = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(\frac{L}{\pi}u)du$, $a_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} g(u)\cos nudu = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(\frac{L}{\pi}u)\cos nudu$ and similarly $b_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} g(u)\sin nudu = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(\frac{L}{\pi}u)\sin nudu.$

Making the change of variable $x = \frac{L}{\pi}u$ we get that

$a_0 = \frac{1}{L}\int\limits_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int\limits_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx$ and

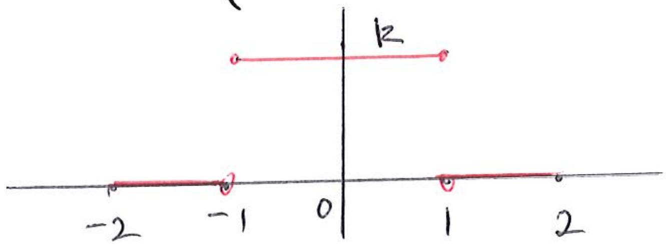
$b_n = \frac{1}{L}\int\limits_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx$, so

$f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty}(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x).$

A piecewise smooth function defined on an interval can be extended periodically to all of $\mathbb{R}$ and then we can apply the theorem to any interval.

Example: Find the Fourier series of the periodic extension of the function

$$f(x) = \begin{cases} 0, & -2 \leq x < -1 \\ k, & -1 \leq x \leq 1 \\ 0, & 1 < x \leq 2 \end{cases}$$

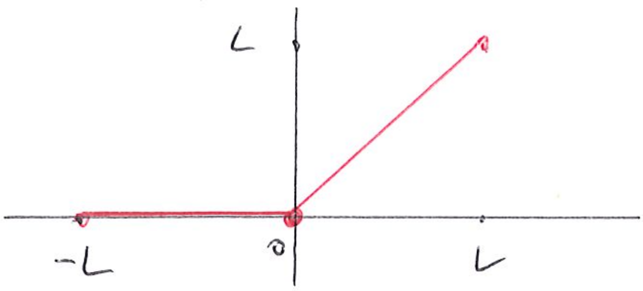


The period is $2L = 4$ so $L = 2$. Then $a_0 = \frac{1}{2}\int\limits_{-2}^{2} f(x)dx = \frac{1}{2}\int\limits_{-1}^{1} kdx = k$, $a_n = \frac{1}{2}\int\limits_{-2}^{2} f(x)\cos\frac{n\pi}{2}xdx = \frac{1}{2}\int\limits_{-1}^{1} k\cos\frac{n\pi}{2}xdx = \frac{2k}{n\pi}\sin\frac{n\pi}{2}$ and $b_n = 0$ for all n since f is an even function.

Therefore $f(x) = \frac{k}{2} + \frac{2k}{\pi}\sum\limits_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\cos\frac{n\pi}{2}x$.

Exercise: Find the Fourier series of the $2L$-periodic extension of the function

$$f(x) = \begin{cases} 0, & -L \leq x < 0 \\ x, & 0 \leq x \leq L \end{cases}$$



Sine and Cosine Series

If a function f is defined on $[0, L]$ only we can extend the definition of f to an even (odd) function on all of $[-L, L]$ and then extend to a $2L$-periodic function on all of $\mathbb{R}$. The even extension of f is given by

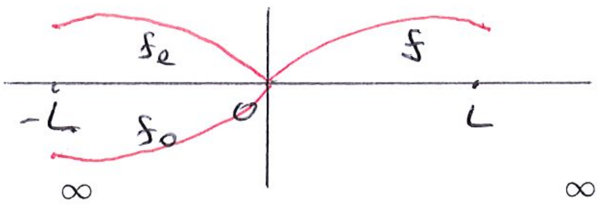
$f_e(x) = f(x)$ for $0 \leq x \leq L$ and $f_e(x) = f(-x)$ for $-L \leq x \leq 0$.

Obviously $f_e(-x) = f_e(x)$ for all $x \in [-L, L]$.

The odd extension of f is given by

$f_o(x) = f(x)$ for $0 \leq x \leq L$ and $f_o(x) = -f(-x)$ for $-L \leq x \leq 0$.

Then we get $f_o(-x) = -f_o(x)$ for all $x \in [-L, L]$.



Now $f_e(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$ and $f_o(x) = \sum\limits_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$ for all $x \in [-L, L]$, where $a_0 = \frac{1}{L} \int\limits_{-L}^{L} f_e(x) dx = \frac{2}{L} \int\limits_{0}^{L} f(x) dx$,

$a_n = \frac{1}{L} \int\limits_{-L}^{L} f_e(x) \cos \frac{n\pi}{L} x dx = \frac{2}{L} \int\limits_{0}^{L} f(x) \cos \frac{n\pi}{L} x dx$ and

$b_n = \frac{1}{L}\int\limits_{-L}^{L} f_o(x)\sin\frac{n\pi}{L}xdx = \frac{2}{L}\int\limits_{0}^{L} f(x)\sin\frac{n\pi}{L}xdx$. In particular this is true for all $x \in [0, L]$, so we have sine and cosine Fourier series for $f(x)$ on $[0, L]$.

Example: Find the Fourier sine and cosine series for $f(x) = x$ on $[0, L]$.

$a_0 = \frac{2}{L}\int\limits_{0}^{L} xdx = L, \quad a_n = \frac{2}{L}\int\limits_{0}^{L} x\cos\frac{n\pi}{L}xdx = \frac{2}{L}[(x\frac{L}{n\pi}\sin\frac{n\pi}{L}x)|_0^L - \int\limits_{0}^{L}\frac{L}{n\pi}\sin\frac{n\pi}{L}xdx] = \frac{2}{L}[\frac{L^2}{n^2\pi^2}\cos\frac{n\pi}{L}]|_0^L = \frac{2L}{n^2\pi^2}(\cos n\pi - 1)$ and

$b_n = \frac{2}{L}\int\limits_{0}^{L} x\sin\frac{n\pi}{L}xdx = \frac{2}{L}[(-x\frac{L}{n\pi}\cos\frac{n\pi}{L}x)|_0^L + \int\limits_{0}^{L}\frac{L}{n\pi}\cos\frac{n\pi}{L}xdx]$
$= -\frac{2L}{n\pi}\cos n\pi$.

Hence the sine series for $f(x)$ is $x = -\frac{2L}{\pi}\sum\limits_{n=1}^{\infty}\frac{\cos n\pi}{n}\sin\frac{n\pi}{L}x$ and the cosine series is $x = \frac{L}{2} + \frac{2L}{\pi^2}\sum\limits_{n=1}^{\infty}\frac{(\cos n\pi - 1)}{n^2}\cos\frac{n\pi}{L}x$.

DIFFERENTIAL EQUATIONS

Ordinary Differential Equations

We first need to briefly consider second order linear ordinary differential equations. A second order linear ordinary differential equations is an equation of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x)$$

or simply $y'' + p(x)y' + q(x)y = r(x)$, where y is a real or complex valued function of a real variable x and $p(x), q(x), r(x)$ are real-valued functions of x. It is homogeneous if $r(x) = 0$. If we insist that $y(x_0) = k_0$ and $y'(x_0) = k_1$ for some x_0 , where k_0 and k_1, are real or complex constants, then we get the

Theorem: If $p(x)$ and $q(x)$ are continuous on some open interval I and $x_0 \in I$, then the initial-value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0, y'(x_0) = k_1$$

has a unique solution $y(x)$ on I.

This is an example of an existence and uniqueness result. We can use any method we like to find this unique solution.

Defining the differential operator L by

$$L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$$

gives the equation in the form $L(y) = r(x)$ with $L(y) = 0$ in the homogeneous case. We have the obvious result, called the principle of superposition:

Theorem: If y_1 and y_2 satisfy $L(y) = 0$, then so does $\alpha y_1 + \beta y_2$ for any real or complex numbers α, β.

Theorem: The solution space of $L(y) = 0$ on the open interval I is two-dimensional.

Proof: Let x_0 be some point in I. Let $y_1(x)$ be the unique solution satisfying $y_1(x_0) = 1$ and $y_1'(x_0) = 0$ and let $y_2(x)$ be the unique solution satisfying $y_2(x_0) = 0$ and $y_2'(x_0) = 1$. Then y_1 and y_2 are linearly independent (consider the Wronskian). Suppose that y is any other solution. Letting $k_1 = y(x_0)$ and $k_2 = y'(x_0)$ we get that y and $k_1y_1 + k_2y_2$ are solutions with the same initial conditions and hence, by uniqueness, are the same. We conclude that y_1 and y_2 form a basis for the solution space.

We shall be interested in the case where $p(x) = a$ and $q(x) = b$ where a and b are real constants i.e.

$$y'' + ay' + by = 0.$$

Recalling that a first order linear differential equation $y'+ky=0$ has a solution $y=e^{-kx}$ we try a solution $y=e^{\lambda x}$ for the second order equation. Substituting into the equation gives $(\lambda^2+a\lambda+b)e^{\lambda x}=0$ and hence $\lambda^2+a\lambda+b=0$. This is called the characteristic equation. Its roots are $\lambda_1=\frac{-a+\sqrt{a^2-4b}}{2}$ and $\lambda_2=\frac{-a-\sqrt{a^2-4b}}{2}$ with corresponding solutions $e^{\lambda_1}x$ and $e^{\lambda_2}x$.

We have three cases:

(1) $a^2-4b>0$, two real roots

(2) $a^2-4b=0$, one real double root

(3) $a^2-4b<0$, two fully complex roots.

In case (1) $y_1=e^{\lambda_1 x}$ and $y_2=e^{\lambda_2 x}$ are linearly independent and so constitute a basis for the solution space on any interval I and the general solution is $y=c_1y_1+c_2y_2$ i.e.

$$y=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}.$$

In case (2) we only get one root $\lambda_1=\lambda_2=-\frac{a}{2}$ and hence only one solution $y_1=e^{-\frac{a}{2}x}$. To find a second linearly independent solution we try a solution of the form $y_2=uy_1$, where $u(x)$ is some function to be determined. Differentiating gives

$$y_2'=u'y_1+uy_1' \text{ and } y_2''=u''y_1+2u'y_1'+uy_1''$$

and now substituting into the equation we get

$$(u''y_1 + 2u'y_1' + uy_1'') + a(u'y_1 + uy_1') + buy_1 = 0.$$

Rearranging gives

$$u''y_1 + u'(2y_1' + ay_1) + u(y_1'' + ay_1' + by_1) = 0.$$

Now the two terms in brackets are 0, so $u''y_1 = 0$ i.e. $u''e^{-\frac{a}{2}x} = 0$. Hence $u'' = 0$ and so $u = d_1x + d_2$ for any constants d_1 and d_2. Taking $d_1 = 1$ and $d_2 = 0$ gives $u(x) = x$. We conclude that $y_2(x) = xy_1(x) = xe^{-\frac{a}{2}x}$ is a second solution. y_1 and y_2 are easily seen to be linearly independent on any interval I and so the general solution is $y = c_1y_1 + c_2y_2$ i.e.

$$y = (c_1 + c_2x)e^{-\frac{a}{2}x}.$$

In case (3) we have $\lambda_1 = -\frac{a}{2} + \frac{\sqrt{a^2-4b}}{2} = -\frac{a}{2} + i\frac{\sqrt{4b-a^2}}{2} = -\frac{a}{2} + i\sqrt{b - \frac{a^2}{4}}$. Writing $\omega^2 = b - \frac{a^2}{4}$ gives roots $\lambda_1 = -\frac{a}{2} + i\omega$ and $\lambda_2 = -\frac{a}{2} - i\omega$. Then $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are complex solutions. Now $e^{s+it} = e^s e^{it} = e^s(\cos t + i\sin t)$, so

$$e^{\lambda_1 x} = e^{-\frac{a}{2}x}(\cos\omega x + i\sin\omega x)$$

$$\text{and } e^{\lambda_2 x} = e^{-\frac{a}{2}x}(\cos\omega x - i\sin\omega x).$$

Now $\frac{1}{2}(e^{\lambda_1 x} + e^{\lambda_2 x})$ and $\frac{1}{2i}(e^{\lambda_1 x} - e^{\lambda_2 x})$ are also solutions i.e. $e^{-\frac{a}{2}x}\cos\omega x$ and $e^{-\frac{a}{2}x}\sin\omega x$ are solutions and are obviously linearly independent. Hence the general solution is

$$y = e^{-\frac{a}{2}x}(A\cos\omega x + B\sin\omega x).$$

Example: Solve the initial-value problem

$$y'' + y' - 2y = 0, \quad y(0) = 4, y'(0) = -5.$$

Characteristic equation is $\lambda^2 + \lambda - 2 = 0$ with roots 1 and -2. General solution is $y = c_1e^x + c_2e^{-2x}$. The initial conditions imply that $c_1 + c_2 = 4$ and $c_1 - 2c_2 = -5$. Hence $c_1 = 1$ and $c_2 = 3$ and the solution is $y = e^x + 3e^{-2x}$.

Example: Solve the initial-value problem

$$y'' - 4y' + 4y = 0, \quad y(0) = 3, y'(0) = 1.$$

Characteristic equation is $\lambda^2 - 4\lambda + 4 = 0$ with a single real root 2. General solution is $y = (c_1 + c_2x)e^{2x}$. The initial conditions imply that $c_1 = 3$ and $2c_1 + c_2 = 1$. Hence $c_1 = 3$ and $c_2 = -5$ and the solution is $y = (3 - 5x)e^{2x}$.

Example: Solve the initial-value problem

$$y'' + 2y' + 5y = 0, \quad y(0) = 1, y'(0) = 5.$$

Characteristic equation is $\lambda^2 + 2\lambda + 5 = 0$ with roots $-1 + 2i$ and $-1 - 2i$. General solution is $y = e^{-x}(A\cos 2x + B\sin 2x)$. The initial conditions imply that $A = 1$ and $-A + 2B = 5$. Hence

$A = 1$ and $B = 3$ and the solution is $y = e^{-x}(\cos 2x + 3\sin 2x)$.

General Partial Differential Equations

A partial differential equation (PDE) is an equation involving partial derivatives. The order of the equation is the highest partial derivative in the equation.

Example: $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$ is first order, $\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial x} = f(x,t)$ is second order etc.

A PDE can have any number of variables ≥ 2.

Example: $u_t = u_{xx}$ has 2 variables, namely t and x, $u_t = u_{rr} + \frac{1}{r}u_r + u_{\theta\theta}$ has 3 variables, namely t, r and θ, etc.

A second order equation in 2 variables is linear if it is of the form $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, where A, B, C, D, E, F and G are functions of x and y i.e. there exists a linear operator L such that $Lu = G$, where

$$L = A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F.$$

Example: $u_{tt} = e^{-x}u_{xx} + \sin t$ (linear)

$$uu_{xx} + u_t = 0 \text{ (non-linear)}$$

$$u_{xx} + yu_{yy} = 0 \text{ (linear)}$$

$$xu_x + yu_y + u^2 = 0 \text{ (non-linear).}$$

We shall be interested in the linear case only. The general linear equation above is homogeneous if $G = 0$. Otherwise it is inhomogeneous.

A solution of a PDE in some region R of the space of the variables involved is a function of the variables that has all the partial derivatives appearing in the equation in some domain containing R and satisfies the equation everywhere in R. Usually there are many such solutions. However if we impose conditions that the solutions must satisfy on the boundary of R (boundary conditions) or, if one of the variables is time t, at $t = 0$ (initial conditions), then hopefully we can get a unique solution. This is our objective. The boundary and initial conditions arise from physical considerations in each particular case.

Important PDEs from physics:

$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$ (one-dimensional heat equation)

$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ (one-dimensional wave equation)

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (two-dimensional Laplace equation)

$\nabla^2 u = 0$ (three-dimensional Laplace equation).

These equations arise in many areas of physics. Our task is to solve these equations given various boundary and initial conditions.

Note: Some PDEs can be solved by integrating.

Example: Solve $u_{xx} - u = 0$, where $u = u(x, y)$.

For each y we consider u as a function of x and use ODE techniques to solve:

$\frac{d^2u}{dx^2} - u = 0$ has characteristic equation $\lambda^2 - 1 = 0$, so any solution has the form $A(y)e^x + B(y)e^{-x}$.

Example: Solve $u_{xy} = -u_x$, where $u = u(x, y)$.

$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial u}{\partial x}$; letting $u_x = p$ gives $\frac{\partial p}{\partial y} = -p$, so $\int \frac{dp}{p} = -\int dy$; hence $\ln p = -y + c$, so $p = De^{-y}$ for each x i.e. $\frac{\partial u}{\partial x} = D(x)e^{-y}$.

We conclude that $u = e^{-y} \int D(x)dx + g(y)$.

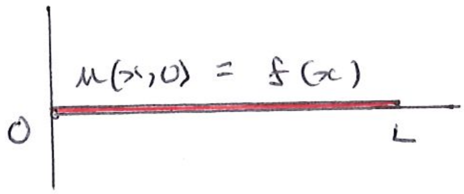
The Heat Equation

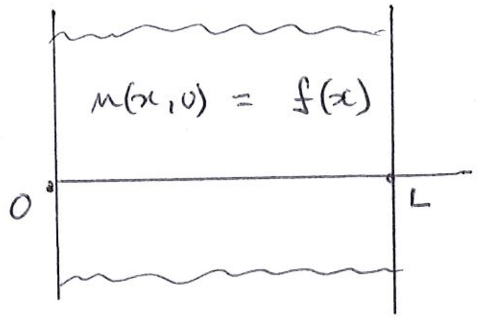
The temperature $u(x,t)$ of a slender metal bar of length L satisfies the diffusion equation

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2},$$

where c is a positive constant. The bar is embedded in a perfect insulator so that the boundary conditions are given by $u(0,t) = 0$ and $u(L,t) = 0$. Initially the temperature of the bar is given by $u(x,0) = f(x)$, for some function $f(x)$. Find the temperature at a distance x from one end of the bar at any time t.

This situation also applies to an infinite vertical slab (see diagram below).





We have

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0$$

with $u(0,t) = 0$, $u(L,t) = 0$ for all t and $u(x,0) = f(x)$ for all x.

To solve we use the method of "Separation of Variables".

Try a solution of the form $u(x,t) = F(x)G(t)$.

Then

$$\frac{\partial u}{\partial t} = F(x)\frac{dG}{dt} \text{ and } \frac{\partial^2 u}{\partial x^2} = \frac{d^2F}{dx^2}G(t),$$

so that
$$F(x)\frac{dG}{dt} = c\frac{d^2F}{dx^2}G(t) \text{ or } \frac{G'}{cG} = \frac{F''}{F},$$

where $'$ means differentiation with respect to the relevant variable. Now the left-hand of this equation is a function of t only while the right-hand side is a function of x only, so both sides must be constant i.e. $\frac{G'}{cG} = \frac{F''}{F} = k$, for some constant k. We get two ODEs

$$\frac{dG}{dt} = ckG \text{ and } \frac{d^2F}{dx^2} = kF$$

or
$$\frac{dG}{dt} - ckG = 0 \text{ and } \frac{d^2F}{dx^2} - kF = 0.$$

Now $u(0,t) = 0$ implies that $F(0)G(t) = 0$ for all t, so $F(0) = 0$ or $G(t) = 0$ for all t. If $G(t) = 0$ for all t, then $u = 0$, the trivial solution (with $f(x) = 0$ also). Hence $F(0) = 0$. In the same way we get $F(L) = 0$.

We now have the second order ODE

$$\frac{d^2F}{dx^2} - kF = 0 \text{ with } F(0) = 0 \text{ and } F(L) = 0.$$

For $k = 0$ the general solution of this initial-value problem is $F(x) = ax + b$. The boundary conditions imply that $a = b =$

0, again the trivial solution. Next consider $k = p^2 > 0$. We have $\frac{d^2F}{dx^2} - p^2F = 0$ with general solution

$$F(x) = Ae^{px} + Be^{-px}$$

The boundary conditions now give $A + B = 0$ and $Ae^{pL} + Be^{-pL} = 0$, so that $B = -A$ and $Ae^{pL} - Ae^{-pL} = 0$. Therefore $A(e^{pL} - \frac{1}{e^{pL}}) = 0$, so $A(\frac{e^{2pL}-1}{e^{pL}}) = 0$ and hence $A = 0$ since $2pL \neq 0$. Again we get the trivial solution. The final possibility is $k = -p^2 < 0$. We have $\frac{d^2F}{dx^2} + p^2F = 0$ with general solution

$$F(x) = A\cos px + B\sin px.$$

Applying the boundary conditions gives $A = 0$ and $B\sin pL = 0$, so $A = 0$ and $B = 0$ or $\sin pL = 0$. If $B = 0$ we again get the trivial solution. Hence $B \neq 0$ and $\sin pL = 0$, so $p = \frac{n\pi}{L}$ for $n \in \mathbb{Z}$. Taking $B = 1$ gives solutions $F_n(x) = \sin\frac{n\pi}{L}x$ for $n = 1, 2, 3...$(For $n < 0$ we get $-\sin\frac{n\pi}{L}x$.) Next consider the other ODE $\frac{dG}{dt} - ckG = 0$, where now $k = -p^2 = -\frac{n^2\pi^2}{L^2}$. For convenience write $\lambda_n = \frac{n\pi}{L}$, so that

$$\frac{dG}{dt} + c\lambda_n^2 G = 0.$$

Try $G = e^{\mu t}$ to get $(\mu + c\lambda_n^2)e^{\mu t} = 0$ or $\mu = -c\lambda_n^2$. We have solutions $G_n = B_n e^{-c\lambda_n^2 t}, n = 1, 2, 3...$

Now for each $n = 1, 2, 3...$ we have a solution $u_n(x,t) = F_n(x)G_n(t) = B_n e^{-c\lambda_n^2 t} \sin \frac{n\pi}{L} x$ for our PDE. However the solution must also satisfy the initial condition $u(x,0) = f(x)$. In general none of the $u_n(x,t)$, or any finite sum of them, will satisfy this condition. Let's try an infinite sum, called a formal sum, leaving aside questions of convergence for the moment,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-c\lambda_n^2 t} \sin \frac{n\pi}{L} x.$$

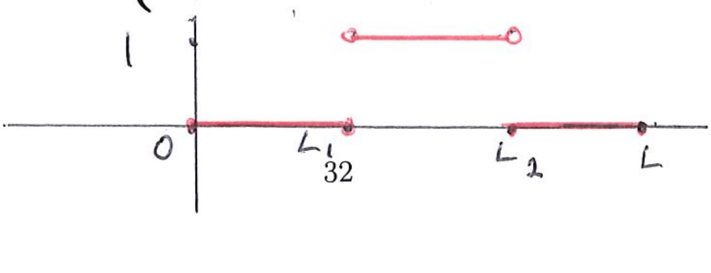
If $u(x,0) = f(x)$, then $f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$ i.e. the B_n must be the coefficients of $f(x)$ in its Fourier sine series expansion on the interval $[0, L]$; in other words

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx.$$

With the B_n so chosen then $u(x,t)$ above will be a solution of the heat equation satisfying all the boundary and initial conditions.

Example:

$$f(x) = \begin{cases} 0, & 0 \le x \le L_1, L_2 \le x \le L \\ 1, & L_1 < x < L_2 \end{cases}$$



$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx = \frac{2}{L}\int_{L_1}^{L_2} 1\sin\frac{n\pi}{L}x\,dx$$
$$= \frac{2}{L}[-\frac{L}{n\pi}\cos\frac{n\pi}{L}x]_{L_1}^{L_2} = \frac{2}{n\pi}(\cos\frac{n\pi L_1}{L} - \cos\frac{n\pi L_2}{L})$$

and so

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos\frac{n\pi L_1}{L} - \cos\frac{n\pi L_2}{L})\sin\frac{n\pi x}{L}e^{-c\lambda_n^2 t}.$$

Note that if $L_1 = \frac{L}{2}$ and $L_2 = L$, then

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos\frac{n\pi}{2} - \cos n\pi)\sin\frac{n\pi x}{L}e^{-c\lambda_n^2 t}.$$

Different boundary conditions for heat equation:

Consider $\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$, $\qquad -L < x < L, \qquad t > 0$

with $u(-L,t) = u(L,t)$, $\frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$ and

$u(x,0) = f(x)$.

As before we write $u(x,t) = F(x)G(t)$ to get

$$\frac{dG}{dt} - ckG = 0 \text{ and } \frac{d^2F}{dx^2} - kF = 0,$$

where k is a constant. The boundary conditions become $F(-L) = F(L)$ and $\frac{dF}{dx}(-L) = \frac{dF}{dx}(L)$. Again, $k = 0$ gives $F(x) = ax + b$ and $F(-L) = F(L)$ implies that $a(-L) + b = a(L) + b$, so $a = 0$ and $F(x) = b$, a constant.

For $k = p^2 > 0$ we have $\frac{d^2F}{dx^2} - p^2F = 0$ with general solution

$$F(x) = Ae^{px} + Be^{-px}.$$

$F(-L) = F(L)$ now gives $Ae^{-pL} + Be^{pL} = Ae^{pL} + Be^{-pL}$, so that $A + Be^{2pL} = Ae^{2pL} + B$ or $(A - B)(1 - e^{2pL}) = 0$. Hence $A = B$ since $p \neq 0$ and $F(x) = A(e^{px} + e^{-px})$. Also $\frac{dF}{dx}(-L) = \frac{dF}{dx}(L)$ gives $Ap(e^{-pL} - e^{pL}) = Ap(e^{pL} - e^{-pL})$, so that $2Ap(e^{-pL} - e^{pL}) = 0$ and hence $A = 0$, the trivial solution. The final possibility is $k = -p^2 < 0$ to give, as before,

$$F(x) = A\cos px + B\sin px.$$

$F(-L) = F(L)$ implies that $A\cos(-pL) + B\sin(-pL) = A\cos pL + B\sin pL$, so $2B\sin pL = 0$. Hence $B = 0$ or $\sin pL = 0$. Also $\frac{dF}{dx}(-L) = \frac{dF}{dx}(L)$ gives $Ap\sin pL + Bp\cos pL = -Ap\sin pL + Bp\cos pL$, so $2Ap\sin pL = 0$. Hence $A = 0$ or $\sin pL = 0$. If $\sin pL \neq 0$ then both A and B are 0 and we get again the trivial solution. We conclude that $\sin pL = 0$ so that $p = \frac{n\pi}{L}, n = 1, 2, 3...$ We have solutions

$$F_n(x) = A_n \cos\tfrac{n\pi}{L}x + B_n \sin\tfrac{n\pi}{L}x, n = 1, 2, 3...$$

and $F_0(x) = A_0$ as the constant solution.

Now for $\frac{dG}{dt} - ckG = 0$. $k = 0$ gives $\frac{dG}{dt} = 0$, so $G =$ a constant. $k = -p^2$ gives solutions $G_n(t) = B_n^* e^{-c\lambda_n^2 t}$, where $\lambda_n = \frac{n\pi}{L}$.

Putting it all together we get solutions

$u_n(x,t) = (A_n \cos\frac{n\pi}{L}x + B_n \sin\frac{n\pi}{L}x)e^{-c\lambda_n^2 t}$ for $n = 0, 1, 2, 3...$;

$n = 0$ giving the constant solution.

Again to satisfy $u(x, 0) = f(x)$ consider

$$u(x,t) = \sum_{n=0}^{\infty}(A_n \cos\tfrac{n\pi}{L}x + B_n \sin\tfrac{n\pi}{L}x)e^{-c\lambda_n^2 t}$$

and so $$f(x) = A_0 + \sum_{n=1}^{\infty}(A_n \cos\tfrac{n\pi}{L}x + B_n \sin\tfrac{n\pi}{L}x).$$

Hence A_0, A_n and B_n must be the Fourier coefficients of $f(x)$ on $[-L, L]$ i.e.

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx, \qquad A_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x dx$$

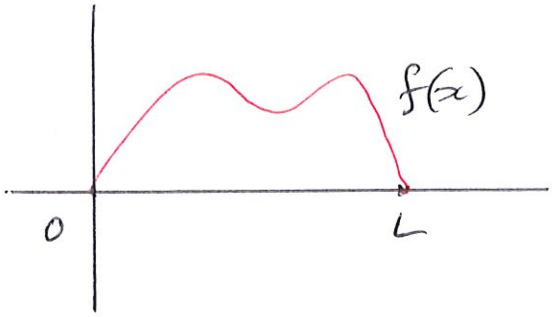
and $B_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x dx$. With these A_0, A_n and B_n we get that $u(x,t)$ is a solution of the equation satisfying the boundary and initial conditions.

The Wave Equation

The deflection of an elastic string of length L fixed at the end-points is governed by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where c is a positive constant. Suppose that the initial deflection is given by $f(x)$ and the initial velocity is given by $g(x)$.



We have

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0$$

with $u(0,t) = 0$, $u(L,t) = 0$ for all t and $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$ for all x. As before try a solution of the form $u(x,t) = F(x)G(t)$.

We get

$$FG'' = c^2 F''G \text{ or } \frac{G''}{c^2 G} = \frac{F''}{F}.$$

Again the only possibility is both sides must be a constant k,

$$\frac{G''}{c^2 G} = \frac{F''}{F} = k.$$

We have two ODEs, namely

$$G'' - c^2kG = 0 \text{ and } F'' - kF = 0.$$

The boundary conditions become $F(0) = 0$ and $F(L) = 0$.

Consider

$$\frac{d^2F}{dx^2} - kF = 0, \text{ with } F(0) = 0 \text{ and } F(L) = 0.$$

This is identical to the heat equation with the only non-trivial solution given by $k = -p^2 < 0$. We have $\frac{d^2F}{dx^2} + p^2F = 0$, with general solution

$$F(x) = A\cos px + B\sin px.$$

Again the boundary conditions give $A = 0$ and $B\sin pL = 0$ so that $p = \frac{n\pi}{L}, n = 1, 2, 3, ...$ Taking $B = 1$ gives $F_n(x) = \sin\frac{n\pi}{L}x$ is a solution for $n = 1, 2, 3, ...$

Now consider

$$\frac{d^2G}{dt^2} + c^2p^2G = 0, \text{ where } p = \frac{n\pi}{L}.$$

Writing $\lambda_n = \frac{cn\pi}{L} > 0$ gives $\frac{d^2G}{dt^2} + \lambda_n^2G = 0$ with general solution

$$G_n(t) = B_n\cos\lambda_nt + B_n^*\sin\lambda_nt,$$

where B_n and B_n^* are constants for each n. We now have $u_n(x,t) = F_n(x)G_n(t) = (B_n\cos\lambda_nt + B_n^*\sin\lambda_nt)\sin\frac{n\pi}{L}x$, $n = 1, 2, 3...$ are solutions of the wave equation satisfying the boundary conditions, called normal modes. To get a solution satisfying the initial conditions also we consider the formal sum

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \tfrac{n\pi}{L} x.$$

Then $f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$, which is true if the B_n are the coefficients of the Fourier sine series expansion of $f(x)$ on the interval $[0,L]$. Similarly $g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi}{L} x$, true if the $B_n^* \lambda_n$ are the coefficients of the Fourier sine series expansion of $g(x)$ on $[0,L]$. In other words, if $B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx$ and $B_n^* \lambda_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x dx$, then $u(x,t)$ is a solution satisfying the initial conditions also.

Note: $u(x,t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi}{L} x + \sum_{n=1}^{\infty} B_n^* \sin \lambda_n t \sin \frac{n\pi}{L} x$
$= \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi}{L} x$, if $g(x) = 0$. In this case
$u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L}(x - ct) + \frac{1}{2} \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L}(x + ct) = \frac{1}{2}((f(x-ct) + f(x+ct))$.

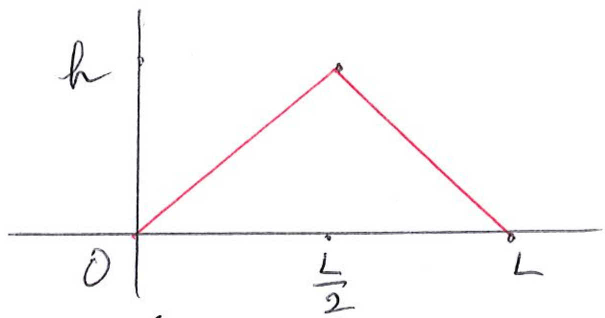
Example: Suppose that $f(x) = \sin 3x - 4 \sin 10x$ and $g(x) = 2 \sin 4x + \sin 6x$ with $L = \pi, c = 2$

Then $\sin 3x - 4 \sin 10x = \sum_{n=1}^{\infty} B_n \sin nx$, so $B_3 = 1, B_{10} = -4$ and all other $B_n = 0$.

Also $2\sin 4x + \sin 6x = \sum\limits_{n=1}^{\infty} \lambda_n B_n^* \sin nx = \sum\limits_{n=1}^{\infty} 2nB_n^* \sin nx$, so $8B_4^* = 2, 12B_6^* = 1$ and all other $B_n^* = 0$. Hence the solution is

$u(x,t) = \cos 6t \sin 3x + \frac{1}{4}\sin 8t \sin 4x + \frac{1}{12}\sin 12t \sin 6x - 4\cos 20t \sin 10x$.

Example: Suppose that the midpoint of the string is pulled up a distance h and then released from rest giving



$$f(x) = \begin{cases} \frac{2h}{L}x, & 0 \le x \le \frac{L}{2} \\ \frac{2h}{L}(L-x), & \frac{L}{2} \le x \le L \end{cases}$$

$$g(x) = 0.$$

$f(x) = \sum\limits_{n=1}^{\infty} B_n \sin \frac{n\pi}{L}x$, where $B_n = \frac{2}{L}\int\limits_0^L f(x)\sin\frac{n\pi}{L}x\,dx =$

$\frac{2}{L}\cdot\frac{2h}{L}\int\limits_0^{\frac{L}{2}} x \sin\frac{n\pi}{L}x\,dx + \frac{2}{L}\cdot\frac{2h}{L}\int\limits_{\frac{L}{2}}^{L}(L-x)\sin\frac{n\pi}{L}x\,dx = \ldots\frac{8h}{\pi^2 n^2}\sin\frac{n\pi}{2}$.

$g(x) = 0$ means that all $B_n^* = 0$.

We have $u(x,t) = \frac{8h}{\pi^2}\sum\limits_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi}{L}x\cos\frac{n\pi c}{L}t$.

d'Alembert's solution of the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{with } t > 0, \quad -\infty < x < \infty$$

and $u(x, 0) = f(x)$, $\frac{\partial u}{\partial t}(x, 0) = g(x)$.

Note that if $\phi : \mathbb{R} \to \mathbb{R}$ is any twice differentiable function and $u(x, t) = \phi(x+ct)$, then $\frac{\partial u}{\partial x} = \phi'(x+ct)$, $\frac{\partial^2 u}{\partial x^2} = \phi''(x+ct)$ and $\frac{\partial u}{\partial t} = c\phi'(x+ct)$, $\frac{\partial^2 u}{\partial t^2} = c^2\phi''(x+ct)$, so that $u(x, t)$ is a solution of the equation. Similarly, if $\psi : \mathbb{R} \to \mathbb{R}$ is any other twice differentiable function and $u(x, t) = \psi(x - ct)$, then $u(x, t)$ is also a solution. We have that $u(x, t) = \phi(x + ct) + \psi(x - ct)$ is a solution. We wish to show that every solution is of this type.

Introduce the variables y, z, where $y = x + ct$ and $z = x - ct$. Then $u = u(y, z)$ giving $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} = c\frac{\partial u}{\partial y} - c\frac{\partial u}{\partial z}$
and $\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) = c\frac{\partial}{\partial y}(\frac{\partial u}{\partial t}) - c\frac{\partial}{\partial z}(\frac{\partial u}{\partial t})$
$= c\frac{\partial}{\partial y}(c\frac{\partial u}{\partial y} - c\frac{\partial u}{\partial z}) - c\frac{\partial}{\partial z}(c\frac{\partial u}{\partial y} - c\frac{\partial u}{\partial z}) = c^2(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 u}{\partial z \partial y} + \frac{\partial^2 u}{\partial z^2})$
$= c^2(\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z^2})$. Similarly $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z^2}$.
Hence $4\frac{\partial^2 u}{\partial y \partial z} = 0$ or $\frac{\partial^2 u}{\partial y \partial z} = 0$. Now $\frac{\partial}{\partial y}(\frac{\partial u}{\partial z}) = 0$ implies that $\frac{\partial u}{\partial z}$ is a function of z only, $h(z)$ say.

Therefore $u = \int h(z)dz + k(y)$, for some function $k(y)$. We write $u(x,t) = \phi(y) + \psi(z) = \phi(x+ct) + \psi(x-ct)$, so every solution is of this form.

Now consider the initial conditions.

$u(x,0) = f(x)$ gives $f(x) = \phi(x) + \psi(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$ gives $g(x) = c(\phi'(x) - \psi'(x))$. Integrating the second identity we get $\phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} g(s)ds$, where x_0 is an arbitrary constant and now solving gives $\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)ds$ and $\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)ds$.

Hence $u(x,t) = \phi(x+ct) + \psi(x-ct)$

$= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)ds + \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)ds$

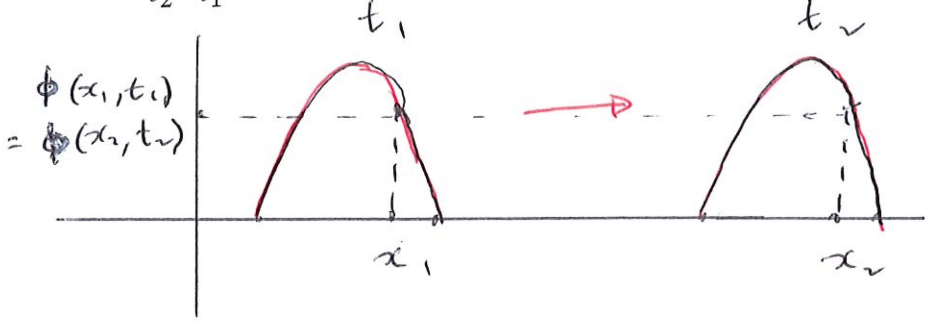
$= \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$

In particular, if $g(x) = 0$, then we get

$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct))$, a superposition of two travelling waves in opposite directions with velocity c.

For a physical interpretation of d'Alembert's solution consider $\phi(x,t) = f(x-ct)$. Suppose that $t_1 < t_2$. ϕ will have the same value at x_1 at time t_1 as at x_2 at time t_2 if $x_1 - ct_1 = x_2 - ct_2$ i.e. $\frac{x_2 - x_1}{t_2 - t_1} = c$.

If x_1 is the space coordinate of any point on the curve $\phi(x,t) = f(x-ct)$ at time t_1, then the same point at time t_2 has coordinate x_2, where $\frac{x_2-x_1}{t_2-t_1} = c$.



Since x_1 is any point on the curve and $t_2 - t_1$ is any time interval $\phi(x,t) = f(x-ct)$ represents a displacement of arbitrary form travelling at constant speed c in the positive x-direction without change of shape. Similarly $\phi(x,t) = f(x+ct)$ represents a displacement of arbitrary form travelling at constant speed c in the negative x-direction without change of shape.

Two Dimensional Laplace's Equation

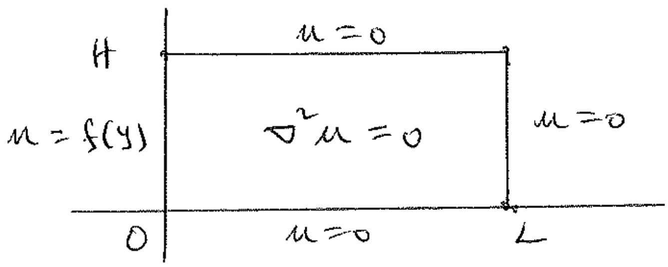
Let $u(x, y)$ be the steady-state temperature in a rectangular metal sheet $\quad 0 \leq x \leq L, \quad 0 \leq y \leq H.$

It is known that $u(x, y)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Suppose the sheet is insulated along the sides $y = 0, y = H$ and $x = L$ and the temperature of the side $x = 0$ is given by $f(y)$. Hence the boundary conditions are

$$u(0, y) = f(y), u(L, y) = 0, u(x, 0) = 0 \text{ and } u(x, H) = 0.$$



Let $u(x, y) = F(x)G(y)$, giving $F''(x)G(y) + F(x)G''(y) = 0$, so $\frac{F''}{F} = -\frac{G''}{G} = k$, a constant. Again we get two ODEs

$$F''(x) - kF(x) = 0, \qquad F(L) = 0$$

and $\qquad G''(y) + kG(y) = 0, \qquad G(0) = 0 \qquad G(H) = 0.$

Consider $G''(y) + kG(y) = 0$. If $k = 0$, then $G(y) = ay + b$. Now $G(0) = 0$ implies $b = 0$ and then $G(H) = 0$ gives $a = 0$, the trivial solution. Next consider $k = -p^2 < 0$. Then

$G''(y) - p^2G(y) = 0$ with general solution

$$G(y) = Ae^{py} + Be^{-py}.$$

$G(0) = 0$ gives $B = -A$, so $G(y) = A(e^{py} - e^{-py})$. Now $G(H) = 0$ implies that $A(e^{2pH} - 1) = 0$, so $A = 0$ also and again we get the trivial solution. The final situation is $k = p^2 > 0$. We have $G''(y) + p^2G(y) = 0$ with general solution

$$G(y) = A\cos py + B\sin py.$$

$G(0) = 0$ gives $A = 0$ and $G(y) = B\sin py$; now $G(H) = 0$ implies that $B = 0$ or $\sin pH = 0$. $B = 0$ again gives the trivial solution, so $\sin pH = 0$ or $p = \frac{n\pi}{H}, n = 1, 2, 3...$ Hence, for each $n = 1, 2, 3...$ we have a solution $G_n(y) = \sin\frac{n\pi}{H}y$.

Now consider $F''(x) - kF(x) = 0$, where $k = p^2 = (\frac{n\pi}{H})^2$. Let $\lambda_n = \frac{n\pi}{H}$. Then we have $F''(x) - \lambda_n^2F(x) = 0$ with general solution

$$F(x) = Ae^{\lambda_n x} + Be^{-\lambda_n x}.$$

$F(L) = 0$ gives $B = -Ae^{2\lambda_n L}$ and we get a solution

$$F_n(x) = A_n(e^{\lambda_n x} - e^{-\lambda_n x}e^{2\lambda_n L})$$

for each $n = 1, 2, 3...$

Rearranging, we get $F_n(x) = A_ne^{\lambda_n L}(e^{\lambda_n x}e^{-\lambda_n L} - e^{-\lambda_n x}e^{\lambda_n L}) =$

$A_n e^{\lambda_n L}(e^{\lambda_n(x-L)} - e^{-\lambda_n(x-L)}) = A_n^{'} \sinh \lambda_n(x-L)$, where $A_n^{'} = 2A_n e^{\lambda_n L}$.

Now for each $n = 1, 2, 3...$ we have a solution

$$u_n(x,y) = A_n^{'} \sinh \lambda_n(x-L) \sin \lambda_n y.$$

None of the $u_n(x,y)$ will, in general, satisfy the boundary condition $u(0,y) = f(y)$, so we consider

$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} A_n^{'} \sinh \lambda_n(x-L) \sin \lambda_n y$. If this satisfies $u(0,y) = f(y)$, then $f(y) = \sum_{n=1}^{\infty} A_n^{'} \sinh(-\lambda_n L) \sin \lambda_n y$, which is true if the $A_n^{'} \sinh(-\lambda_n L)$ are the Fourier sine series coefficients of $f(y)$ on $[0, H]$ i.e.

$$A_n^{'} \sinh(-\lambda_n L) = \frac{2}{H} \int_0^H f(y) \sin \frac{n\pi}{H} y dy.$$

Example: Suppose $f(y) = y(H-y)$.

$\int_0^H y(H-y) \sin \frac{n\pi}{H} y dy = H \int_0^H y \sin \frac{n\pi}{H} y dy - \int_0^H y^2 \sin \frac{n\pi}{H} y dy.$

Let $I = H \int_0^H y \sin \frac{n\pi}{H} y dy$ and $J = \int_0^H y^2 \sin \frac{n\pi}{H} y dy$.

Integration by parts gives $I = H[-\frac{H}{n\pi} y \cos \frac{n\pi}{H} y]_0^H + \frac{H}{n\pi} \int_0^H \cos \frac{n\pi}{H} y dy$

$= -\frac{H^3}{n\pi} \cos n\pi.$

Also $J = [-\frac{H}{n\pi} y^2 \cos \frac{n\pi}{H} y]_0^H + \frac{H}{n\pi} \int_0^H 2y \cos \frac{n\pi}{H} y dy$

$= -\frac{H^3}{n\pi}\cos n\pi + \frac{2H^2}{n^2\pi^2}[y\sin\frac{n\pi}{H}y]_0^H - \frac{2H^2}{n^2\pi^2}\int\limits_0^H \sin\frac{n\pi}{H}y dy =$

$-\frac{H^3}{n\pi}\cos n\pi + \frac{2H^3}{n^3\pi^3}[\cos\frac{n\pi}{H}y]_0^H = -\frac{H^3}{n\pi}\cos n\pi + \frac{2H^3}{n^3\pi^3}(\cos n\pi - 1).$

Hence $A_n^{'}\sinh(-\lambda_n L) = \frac{4H^2}{n^3\pi^3}(1-\cos n\pi)$, so $A_n^{'} = \frac{4H^2}{n^3\pi^3}\frac{(1-\cos n\pi)}{\sinh(-\lambda_n L)}$

and $u(x,y) = \frac{4H^2}{\pi^3}\sum\limits_{n=1}^{\infty}\frac{(\cos n\pi - 1)}{n^3\sinh(\lambda_n L)}\sinh\lambda_n(x-L)\sin\lambda_n y.$

Note: Consider

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 \le x \le L, \qquad 0 \le y \le H,$$

where $u(x,0) = f_1(x)$, $u(x,H) = f_2(x)$, $u(0,y) = g_1(y)$ and $u(L,y) = g_2(y)$.

Separation of variables depends on some bounday conditions being homogeneous i.e. $= 0$. To solve the above we consider solutions of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

with $u(x,0) = f_1(x)$ and the others $= 0$ etc. If the solutions are u_1, u_2, u_3, u_4, respectively, then $u = u_1 + u_2 + u_3 + u_4$ will be a solution satisfying all the boundary conditions above.

Example: The voltage $V(x,y)$ at any point in a square metal plate of side length π satisfies Laplace'e equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

The plate is earthed at $x = 0, x = \pi$ and $y = 0$, so that

$V(0, y) = 0, V(\pi, y) = 0$ and $V(x, 0) = 0$. A voltage $f(x)$ is applied along the fourth side $y = \pi$ so that $V(x, \pi) = f(x)$, where

$$f(x) = \begin{cases} x, & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

Solve for $V(x, y)$.

$V(x, y) = F(x)G(y)$ so that $F''(x) - kF(x) = 0$ and $G''(y) + kG(y) = 0$, with $F(0) = 0, F(\pi) = 0$ and $G(0) = 0$.

Consider $F''(x) - kF(x) = 0$, with $F(0) = 0, F(\pi) = 0$. $k = 0$ implies $F = 0$. $k = p^2$ also gives $F = 0$ as before. Therefore $k = -p^2$ and so $F''(x) + p^2F(x) = 0$, with general solution

$$F(x) = A\cos px + B\sin px.$$

The boundary conditions now give $A = 0$ and $\sin p\pi = 0$ so that $p = n, n = 1, 2, 3, ...$ and $\sin nx$ is a solution for each $n = 1, 2, 3...$

Now consider $G''(y) - p^2G(y) = 0$, with $G(0) = 0, p = n$.

For each n we have a general solution

$$G_n(y) = A_n e^{ny} + B_n e^{-ny}.$$

$G(0) = 0$ implies that $B_n = -A_n$ so $G_n(y) = A_n(e^{ny} - e^{-ny})$.

We have solutions $V_n(x, y) = A_n \sin nx(e^{ny} - e^{-ny}) = A_n^{'} \sin nx \sinh ny$.

Consider $V(x,y) = \sum\limits_{n=1}^{\infty} A_n^{'} \sin nx \sinh ny$. Then

$V(x,\pi) = f(x)$ implies that $f(x) = \sum\limits_{n=1}^{\infty} A_n^{'} \sin nx \sinh n\pi$, so that $A_n^{'} \sinh n\pi$ is the Fourier coefficient of the sine series expansion of $f(x)$ on the iterval $[0,\pi]$ i.e. $A_n^{'} \sinh n\pi =$

$\frac{2}{\pi}\int\limits_0^{\pi} f(x) \sin nx dx = \frac{2}{\pi}[\int\limits_0^{\frac{\pi}{2}} x \sin nx dx + \pi \int\limits_{\frac{\pi}{2}}^{\pi} \sin nx dx - \int\limits_{\frac{\pi}{2}}^{\pi} x \sin nx dx]$

$= \frac{4}{\pi n^2} \sin \frac{n\pi}{2}$, on integrating.

Finally we have $V(x,y) = \frac{4}{\pi} \sum\limits_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n^2 \sinh n\pi} \sin nx \sinh ny$.

Example: The voltage $V(x,y)$ at any point in a square metal plate of side length 2π satisfies Laplace'e equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

The plate is earthed at $x = 0, x = 2\pi$ and $y = 2\pi$, so that $V(0,y) = 0, V(2\pi,y) = 0$ and $V(x,2\pi) = 0$. A voltage $\sin 2x$ is applied along the fourth side $y = 0$ so that $V(x,0) = \sin 2x$. Solve for $V(x,y)$.

$V(x,y) = F(x)G(y)$ so that $F^{''}(x) - kF(x) = 0$ and $G^{''}(y) + kG(y) = 0$, with $F(0) = 0, F(2\pi) = 0$ and $G(2\pi) = 0$.

Consider $F^{''}(x) - kF(x) = 0$, with $F(0) = 0, F(2\pi) = 0$. $k = 0$ implies $F = 0$. $k = p^2$ also gives $F = 0$ as before.

Therefore $k = -p^2$ and so $F''(x) + p^2 F(x) = 0$, with general solution

$$F(x) = A\cos px + B\sin px.$$

The boundary conditions now give $A = 0$ and $\sin 2p\pi = 0$ so that $p = \frac{n}{2}, n = 1, 2, 3, ...$ and $\sin\frac{n}{2}x$ is a solution for each $n = 1, 2, 3...$

Now consider $G''(y) - p^2 G(y) = 0$, with $G(2\pi) = 0, p = \frac{n}{2}$.

For each n we have a general solution

$$G_n(y) = A_n e^{\frac{n}{2}y} + B_n e^{-\frac{n}{2}y}.$$

$G(2\pi) = 0$ now gives $A_n e^{n\pi} + B_n e^{-n\pi} = 0$, so $B_n = -A_n e^{2n\pi}$.

Therefore $G_n(y) = A_n(e^{\frac{n}{2}y} - e^{2n\pi}e^{-\frac{n}{2}y})$

$= A_n e^{n\pi}(e^{(\frac{n}{2}y - n\pi)} - e^{-(\frac{n}{2}y - n\pi)}) = 2A_n e^{n\pi}\sinh n(\frac{y}{2} - \pi)$.

Write $G_n(y) = A_n' \sinh n(\frac{y}{2} - \pi)$ and then

$V_n(x, y) = A_n' \sinh n(\frac{y}{2} - \pi)\sin\frac{n}{2}x$ is a solution for each $n = 1, 2, 3...$ Now consider $V(x, y) = \sum_{n=1}^{\infty} A_n' \sinh n(\frac{y}{2} - \pi)\sin\frac{n}{2}x$.

If this is a solution, then $V(x, 0) = \sin 2x$ implies that

$\sin 2x = \sum_{n=1}^{\infty} A_n' \sinh n(-\pi)\sin\frac{n}{2}x$. We conclude that

$A_4' \sinh(-4\pi) = 1$ and all other $A_n' = 0$. We get our solution

$V(x, y) = \frac{1}{\sinh(-4\pi)}\sinh 4(\frac{y}{2} - \pi)\sin 2x = \frac{\sinh(4\pi - 2y)}{\sinh 4\pi}\sin 2x$.

FOURIER TRANSFORM

Definition of Fourier Transform

Consider $f : [-L, L] \to \mathbb{R}$ or $f : \mathbb{R} \to \mathbb{R}$ periodic with period $2L$. Then

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x)$, where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$,

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x dx$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x dx$. Now

$e^{i\frac{n\pi}{L}x} = \cos\frac{n\pi}{L}x + i\sin\frac{n\pi}{L}x$ and $e^{-i\frac{n\pi}{L}x} = \cos\frac{n\pi}{L}x - i\sin\frac{n\pi}{L}x$,

so that $\cos\frac{n\pi}{L}x = \frac{1}{2}(e^{i\frac{n\pi}{L}x} + e^{-i\frac{n\pi}{L}x})$ and

$\sin\frac{n\pi}{L}x = \frac{1}{2i}(e^{i\frac{n\pi}{L}x} - e^{-i\frac{n\pi}{L}x})$. Hence

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(\frac{(a_n - ib_n)}{2}e^{i\frac{n\pi}{L}x} + \frac{(a_n+ib_n)}{2}e^{-i\frac{n\pi}{L}x}) =$

$c_0 + \sum_{n=1}^{\infty}(c_n e^{i\frac{n\pi}{L}x} + c_{-n}e^{-i\frac{n\pi}{L}x})$, where $c_0 = \frac{a_0}{2}$, $c_n = \frac{a_n - ib_n}{2}$ and

$c_{-n} = \frac{a_n+ib_n}{2}$. Therefore $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x}$, where for each

n, $c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi}{L}x}dx$.

Now set $\frac{n\pi}{L} = \omega_n$, so that $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x} =$

$\frac{1}{2L}\sum_{n=-\infty}^{\infty}[\int_{-L}^{L} f(t)e^{-i\omega_n t}dt]e^{i\omega_n x} = \frac{1}{2L}\sum_{n=-\infty}^{\infty} e^{i\omega_n x}\int_{-L}^{L} f(t)e^{-i\omega_n t}dt$.

Let $\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$ and consider $L \to \infty$.

Then $f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{i\omega_n x}(\int_{-L}^{L} f(t)e^{-i\omega_n t}dt)\Delta\omega$. Letting

$L \to \infty$, so that $\Delta\omega \to 0$ and $\omega_n \to$ a continuous variable ω,

to give $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} (\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt) d\omega =$
$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$, where $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$ is called the Fourier transform of f and $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$ is called the inverse Fourier transform of $\hat{f}$. We have

$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$ and $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$.

All of the above is formalised in the following theorem.

Theorem: If $f(x)$, $-\infty < x < \infty$, is piecewise continuous on each finite interval and if $\int_{-\infty}^{\infty} |f(x)| dx < \infty$, then the Fourier transform $\hat{f}(\omega)$ exists. Furthermore

$$\frac{1}{2}(f(x^+) + f(x^-)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega.$$

Properties of Fourier Transform

(i) $\widehat{f+g}(\omega) = \hat{f}(\omega) + \hat{g}(\omega)$ and $\widehat{af}(\omega) = a\hat{f}(\omega)$.

Proof: Obvious from the definition.

(ii) If $f(x)$ is continuous on $\mathbb{R}$ and $f(x) \to 0$ as $|x| \to \infty$ and $\int\limits_{-\infty}^{\infty} |f'(x)|dx < \infty$, then $\hat{f'}(\omega) = i\omega\hat{f}(\omega)$.

Proof: $\hat{f'}(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f'(x)e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}[f(x)e^{-i\omega x}]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}}(-i\omega) \int\limits_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = 0 + \frac{(i\omega)}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = i\omega\hat{f}(\omega)$.

Note that from this we can deduce that $\hat{f''}(\omega) = (i\omega)(i\omega)\hat{f}(\omega) = -\omega^2\hat{f}(\omega)$.

(iii) $\int\limits_a^b f(x)dx = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \hat{f}(\omega)(\frac{e^{i\omega b} - e^{i\omega a}}{i\omega})d\omega$.

Proof: $\int\limits_a^b f(x)dx = \frac{1}{\sqrt{2\pi}} \int\limits_a^b [\int\limits_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega]dx$

$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} [\int\limits_a^b \hat{f}(\omega)e^{i\omega x}dx]d\omega = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} [\hat{f}(\omega) \int\limits_a^b e^{i\omega x}dx]d\omega$

$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \hat{f}(\omega)(\frac{e^{i\omega b} - e^{i\omega a}}{i\omega})d\omega$.

Examples

(i)

$$f(x) = \begin{cases} k, & 0 \le x \le a \\ 0, & x < 0 \text{ or } x > a \end{cases}$$

$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_0^a k e^{-i\omega x} dx; \quad \omega = 0$ gives $\hat{f}(\omega) = \frac{ka}{\sqrt{2\pi}}$.

$\omega \neq 0$ gives $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_0^a k e^{-i\omega x} dx = \frac{k}{\sqrt{2\pi}} [\frac{e^{-i\omega x}}{-i\omega}]_0^a = \frac{k}{\sqrt{2\pi}} [\frac{1 - e^{-i\omega a}}{i\omega}]$.

(ii)

$$f(x) = \begin{cases} 1, & a \le x \le b \\ 0, & x < a \text{ or } x > b \end{cases}$$

$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_a^b e^{-i\omega x} dx; \quad \omega = 0$ gives $\hat{f}(\omega) = \frac{b-a}{\sqrt{2\pi}}$.

$\omega \neq 0$ gives $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_a^b e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} [\frac{e^{-i\omega x}}{-i\omega}]_a^b = \frac{1}{\sqrt{2\pi}} [\frac{e^{-i\omega a} - e^{-i\omega b}}{i\omega}]$.

(iii)

$$f(x) = \begin{cases} e^x, & x \le 0 \\ 0, & x > 0 \end{cases}$$

$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^0 e^x e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^0 e^{(1-i\omega)x} dx = \frac{1}{\sqrt{2\pi}} [\frac{e^{(1-i\omega)x}}{1-i\omega}]_{-\infty}^0 =$
$\frac{1}{\sqrt{2\pi}(1-i\omega)}$.

(iv)

$$f(x) = \begin{cases} xe^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty xe^{-x}e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty xe^{-(1+i\omega)x}dx =$

$\frac{1}{\sqrt{2\pi}}[\frac{xe^{-(1+i\omega)x}}{-(1+i\omega)}]_0^\infty - \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty \frac{e^{-(1+i\omega)x}}{-(1+i\omega)}dx = \frac{1}{\sqrt{2\pi}}[\frac{e^{-(1+i\omega)x}}{-(1+i\omega)^2}]_0^\infty = \frac{1}{\sqrt{2\pi}(1+i\omega)^2}.$

(v)

$$f(x) = \begin{cases} k, & -\pi \leq x \leq \pi \\ 0, & x < -\pi, \quad \pi < x \end{cases}$$

$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^\infty f(x)e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}} \int\limits_{-\pi}^\pi f(x)e^{-i\omega x}dx$

$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\pi}^\pi f(x)(\cos\omega x - i\sin\omega x)dx = \frac{2}{\sqrt{2\pi}} \int\limits_0^\pi f(x)\cos\omega x dx$

(since $f(x)$ is even) $= \frac{2k}{\sqrt{2\pi}} \int\limits_0^\pi \cos\omega x dx = \frac{2k}{\sqrt{2\pi}} \frac{\sin \pi\omega}{\omega}.$

(vi) The truncated cos function

$$f(x) = \begin{cases} \cos 3x, & -\pi \leq x \leq \pi \\ 0, & x < -\pi, \quad \pi < x \end{cases}$$

$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^\infty f(x)e^{-i\omega x}dx = \frac{2}{\sqrt{2\pi}} \int\limits_0^\pi \cos 3x \cos\omega x dx$

(as in (v)) $= \frac{2\omega \sin \pi\omega}{\sqrt{2\pi}(9-\omega^2)}.$

The Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0, \quad c > 0.$$

Assume that u and $\frac{\partial u}{\partial x}$ are finite as $|x| \to \infty$ and

$u(x,0) = f(x), \quad -\infty < x < \infty$, where $f(x)$ is piecewise smooth on every finite subinterval and $\int\limits_{-\infty}^{\infty} |f(x)|dx$ is finite.

Define the spatial Fourier transform of $u(x,t)$ to be

$\hat{u}(\omega, t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx.$

Applying the spatial transform to the differential equation we get $\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x,t)e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} c^2 \frac{\partial^2 u}{\partial x^2}(x,t)e^{-i\omega x}dx,$

so that $\frac{\partial}{\partial t}[\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx] = c^2 \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}(x,t)e^{-i\omega x}dx$

giving $\frac{\partial \hat{u}}{\partial t}(\omega, t) = -c^2\omega^2\hat{u}(\omega,t)$ and hence $\hat{u}(\omega,t) = A(\omega)e^{-c^2\omega^2 t}$,

where $A(\omega)$ is some function of ω.

Now $u(x,0) = f(x)$ implies that $\hat{u}(\omega, 0) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u(x,0)e^{-i\omega x}dx =$

$\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \hat{f}(\omega)$, so

$A(\omega) = \hat{f}(\omega)$ and $\hat{u}(\omega,t) = \hat{f}(\omega)e^{-c^2\omega^2 t}$. Taking the inverse Fourier transform we get $u(x,t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \hat{f}(\omega)e^{-c^2\omega^2 t}e^{i\omega x}d\omega =$

$\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \hat{f}(\omega)e^{\omega(ix - c^2\omega t)}d\omega.$

The Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0, \quad c > 0.$$

Assume that u and $\frac{\partial u}{\partial x}$ are finite as $|x| \to \infty$ and $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$, $-\infty < x < \infty$, where $f(x)$ and $g(x)$ are piecewise smooth on every finite subinterval and $\int_{-\infty}^{\infty} |f(x)|dx$, $\int_{-\infty}^{\infty} |g(x)|dx$ are both finite.

Applying the spatial transform to the differential equation we get $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2}(x,t)e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c^2 \frac{\partial^2 u}{\partial x^2}(x,t)e^{-i\omega x}dx$, so that $\frac{\partial^2}{\partial t^2}[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx] = c^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}(x,t)e^{-i\omega x}dx$ giving $\frac{\partial^2 \hat{u}}{\partial t^2}(\omega,t) = -c^2\omega^2\hat{u}(\omega,t)$ and hence $\frac{\partial^2 \hat{u}}{\partial t^2}(\omega,t) + c^2\omega^2\hat{u}(\omega,t) = 0$. The general solution is

$$\hat{u}(\omega,t) = A(\omega)e^{ic\omega t} + B(\omega)e^{-ic\omega t}.$$

Now $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$ imply that $\hat{u}(\omega,0) = \hat{f}(\omega)$ and $\frac{\partial \hat{u}}{\partial t}(\omega,0) = \hat{g}(\omega)$, so that $A(\omega) + B(\omega) = \hat{f}(\omega)$ and $ic\omega(A(\omega) - B(\omega)) = \hat{g}(\omega)$. Hence $A(\omega) = \frac{1}{2}(\hat{f}(\omega) + \frac{\hat{g}(\omega)}{ic\omega})$ and $B(\omega) = \frac{1}{2}(\hat{f}(\omega) - \frac{\hat{g}(\omega)}{ic\omega})$ and so $u(x,t) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} [(\hat{f}(\omega)+\frac{\hat{g}(\omega)}{ic\omega})e^{ic\omega t}e^{i\omega x}+(\hat{f}(\omega)-\frac{\hat{g}(\omega)}{ic\omega})e^{-ic\omega t}e^{i\omega x}]d\omega$

$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega(x+ct)}d\omega + \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega(x-ct)}d\omega +$

$\frac{1}{2\sqrt{2\pi}c}\int\limits_{-\infty}^{\infty}\hat{g}(\omega)(\frac{e^{i\omega(x+ct)}-e^{i\omega(x-ct)}}{i\omega})d\omega = \frac{1}{2}[f(x+ct)+f(x-ct)]+$

$\frac{1}{2c}\int\limits_{x-ct}^{x+ct} g(v)dv$, which is d'Alembert's solution.

(Recall property (iii) of the Fourier transform.)

Considering the simpler case of $g(x) = 0$, we get

$A(\omega) + B(\omega) = \hat{f}(\omega)$ and $ic\omega(A(\omega) - B(\omega)) = 0$. Hence

$A(\omega) = \frac{1}{2}\hat{f}(\omega)$ and $B(\omega) = \frac{1}{2}\hat{f}(\omega)$ and so

$u(x,t) = \frac{1}{2\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}[\hat{f}(\omega)e^{ic\omega t}e^{i\omega x} + \hat{f}(\omega)e^{-ic\omega t}e^{i\omega x}]d\omega$

$= \frac{1}{2\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega(x+ct)}d\omega + \frac{1}{2\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega(x-ct)}d\omega$

$= \frac{1}{2}[f(x+ct)+f(x-ct)]$ which again is d'Alembert's solution.