# General Relativity

MA4448

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# Contents

# Suggested Reading

R. D'Inverno, *Introducing Einstein's Relativity* (+ - - - )

B. Shutz, *A First Course in General Realtivity* (- + + +)

S.M.Carroll, *Lecture Notes on GR*, online (- + + +)

H. Stephani, *General Relativity* (+ - - -)

R.M. Wald, *General Relativity* (- + + +)

C.W.Misner, K.S. Throne, J.A.Wheeler, *Gravitation* (- + + +)

# 1 Introduction

## 1.1 Newtonian Theory of Gravity

For a distribution of matter with density $\rho(t, x, y, z)$, we have a gravitational potential $\varphi$.

Field Equation:

$$\nabla^2 \varphi = 4\pi G \rho$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$\varphi$ is given at points inside the matter distribution by solving Poisson's equation with $\rho$ given.

Outside the matter distribution (i.e. in a vacuum) $\rho$ is given by solving

$$\nabla^2 \varphi = 0$$

## Equations of Motion

The equations of motion of a test particle are

$$x^i = x^i(t), i = 1, 2, 3$$

$$\Rightarrow \frac{d^2 x^i}{dt^2} = -\varphi_{,i}$$

i.e.

$$\frac{d^2 x}{dt^2} = -\frac{\partial \varphi}{\partial x}$$
$$\frac{d^2 y}{dt^2} = -\frac{\partial \varphi}{\partial y}$$
$$\frac{d^2 z}{dt^2} = -\frac{\partial \varphi}{\partial z}$$

Newtonian theory can be written as a metric theory of gravity, though the spacetime is complicated to describe geometrically. (Requires additional structures such as absolute time and simultaneous points forming a Euclidean 3-manifold)

E.g. We can rewrite the equations of motion in the geodesic equation form.

**Define**

$$
\begin{aligned}
x^\mu &= (t, x, y, z) \\
\Rightarrow \dot{x}^\mu &= (1, \dot{x}^i) \\
\ddot{x}^\mu &= (0, \ddot{x}^i) = (0, \varphi_{,i})
\end{aligned}
$$

i.e.

$$\ddot{t} = 0 \text{ and } \ddot{x}^i + \varphi_{,i} = 0$$

$$\Rightarrow \ddot{x}^i + \varphi_{,i}\dot{x}^0\dot{x}^0 = 0$$

Compare with the geodesic equation

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt} = 0$$

We can read off the Newtonian connection

$$\Gamma^i{}_{N00} = \varphi_{,i} \text{ and } \Gamma^\mu{}_{N\nu\lambda} = 0 \text{ otherwise}$$

$$\Rightarrow R_N{}^i_{0j0} = -\varphi_{,i} \text{ and } R_N{}^\mu_{\nu\lambda\rho} = 0 \text{ otherwise}$$

$$\Rightarrow R_{N00} = \nabla^2\varphi \text{ and } R_{N\mu\nu} = 0 \text{ otherwise}$$

i.e. Newtonian spacetime is curved.

## 1.2 Special Relativity

Describes non gravitational physics such as electrodynamics, standard model etc.

Discard the notion of absolute time, we introduce a 4D space continuum. To each event in spacetime, we assign the coordinates $(t, x, y, z)$ and the infinitesimal interval $ds$ between the infinitesimally separated events satisfies the Minkowski line element.

$$
\begin{aligned}
ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\
&= \eta_{\mu\nu}dx^\mu dx^\nu
\end{aligned}
$$

where

$$\eta_{\mu\nu} = diag\,(-1, 1, 1, 1)$$

The Minkowski line element is invariant under Lorentz transformations

$$x^\mu \to x^{\mu'}$$

where

$$\Lambda_\mu{}^{\mu'}\Lambda_\nu{}^{\nu'}\eta_{\mu'\nu'} = \eta_{\mu\nu}$$

and under arbitrary translations

$$x^\mu \to x^\mu + d^\mu$$

where $d^\mu$ are constants. Together, these define the Poincare group.

Null cones of $\nu_{\mu\nu}$ describe light rays in a vacuum.
Time-like geodesics of $\nu_{\mu\nu}$ describe force free motion of massive particles and

$$\tau = \int d\tau = \int(-\eta_{\mu\nu}dx^\mu dx^\nu)^{\frac{1}{2}} = \int(1-v^2)^{\frac{1}{2}}dt$$

is the proper time measured by a standard clock associated with the particle where the integral is taken along a time like path representing the particle's trajectory.

**Note** Used 'relativistic' units $c = 1$. In non-relativistic units

$$\tau = \int \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} dt$$

## 1.3 General Relativitiy (Guiding Principles)

- All observers are equivalent.

  $\Rightarrow$ physics should be coordinate independent. (Principle of General Covariance)
  $\Rightarrow$ tensor equations are the most natural mathematical framework.

- Should agree locally with SR

- Admit a class of preferred relatively accelerated world lines representing free fall.

- Should admit a tensor related to the source of the gravitational field.

- Should explain observed solar system phenonema such as light deflection, perihelian advance of Mercury, time-delay etc.

General relativity assumes spacetime is a pseudo-riemannian manifold with signature(- + + +).

Null geodesics represent light rays.

Timelike geodesics represent paths of freely falling particles.

Locally, we can always choose a chart s.t. $g_{\mu\nu} = \eta_{\mu\nu}$, i.e. SR valid locally. The field equations are

$$G^{\mu\nu} = \kappa T^{\mu\nu}$$

where $\kappa$ is a constant fixed by the Newtonian limit, $G^{\mu\nu}$ is the Einstein Tensor, and $T^{\mu\nu}$ describes the source of the gravitational field.

# 2 Einstein Equations from an Action Principle

We first recall two important results:
(i) Fundamental Lemma of Calculus of Variations: If

$$\int_{x_1}^{x_2} \varphi(x)\eta(x)dx = 0$$

where $\varphi(x)$ is continuous, ad $\eta(x)$ twice differentiable and vanishes on boundary $\eta(x_1) = \eta(x_2) = 0$, then $\varphi(x) = 0$ on $[x_1, x_2]$
(ii) Gauss Divergence Theorem:

$$\int_V \nabla_\mu \chi^\mu d\Omega = \int_{\partial V} \chi^\mu d\Sigma_\mu$$

where $\chi^\mu$ is a vector density of weight 1. An immediate corollary is

$$\int_V \sqrt{-g}\, \nabla_\mu X^\mu d\Omega = \int_{\partial V} \sqrt{-g}\, X^\mu d\Sigma_\mu$$

where $X^\mu$ is a vector field.

## 2.1 Principle of Least Action

We start with an action

$$S = \int_{\text{all space}} \mathcal{L} d\Omega$$

where $\mathcal{L}$ is a Langrangian density of weight 1. We consider small variations in the metric tensor $g_{\mu\nu} \to g_{\mu\nu} + S g_{\mu\nu}$ which inducs a variation in the action functional $S \to S + \delta S$. We also assume the metric variations and its derivatives vanish at infinity.

The action principle implies

$$\delta S = \int_{\text{all space}} \mathcal{L}^{\mu\nu}\delta g_{\mu\nu} d\Omega = 0$$

where $\mathcal{L}^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}}$ is a $\binom{2}{0}$ tensor density of weight 1.

## 2.2 The Stress-Energy-Momentum Tensor

In General Relativity, we must allow for the definition of a tensor related to the source of the gravitational field, i.e. the action has contributions coming from the matter fields and the gravitational fields

$$S = S_\mu + S_g = \int_{\text{all space}} (\mathcal{L}_\mu + \mathcal{L}_g) d\Omega$$

We define

$$\delta S_\mu = \int_{\text{all space}} \frac{\delta \mathcal{L}_\mu}{\delta g_{\mu\nu}} \delta g_{\mu\nu} d\Omega = \frac{1}{2}\int_{\text{all space}} \sqrt{-g}\, T^{\mu\nu}\delta g_{\mu\nu} d\Omega$$

where we have defined

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta \mathcal{L}_\mu}{\delta g_{\mu\nu}}$$

which is the stress-energy-momentum tensor.

## 2.3 Varying the Metric Inverse and the Metric Determinant

In what follows, we shall require $\delta g^{\mu\nu}$ in terms of $\delta g_{\mu\nu}$. We note that

$$g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\lambda$$

$$\Rightarrow \delta g^{\mu\nu} g_{\nu\lambda} + g^{\mu\nu}\partial g_{\nu\lambda} = 0$$

Multiply by $g^{\lambda\rho}$

$$\Rightarrow \delta g^{\mu\nu}\delta^\rho_\nu = -g^{\lambda\rho} g^{\mu\nu}\delta g_{\nu\lambda}$$

$$\boxed{\Rightarrow \delta g^{\mu\rho} = -g^{\lambda\rho} g^{\mu\nu}\delta g_{\nu\lambda}} \qquad (2.1)$$

We also require $\delta\sqrt{-g}$. We note that for any non-singular matrix $a_{\mu\nu}$, with inverse $a^{\mu\nu}$ and determinant $a$. Each element has a cofactor given by

$$A^{\mu\nu} = a^{\nu\mu} a$$

Also, the determinant is obtained by expanding across any row.

$$a = A^{(\mu)\nu} a_{(\mu)\nu} \qquad \text{(no sum over } \mu)$$

which imples that

$$\frac{\partial a}{\partial a_{\mu\nu}} = A^{\mu\nu} = a^{\nu\mu} a$$

Therefore

$$\partial a = \frac{\partial a}{\partial a_{\mu\nu}} \partial a_{\mu\nu} = a^{\nu\mu} a \partial a_{\mu\nu}$$

$a = \sqrt{-g}$, this gives

$$\partial(\sqrt{-g}) = \frac{1}{2}(-g)^{\frac{1}{2}} \partial g$$

$$= -\frac{1}{2}(-g)^{\frac{1}{2}} g g^{\nu\mu} \partial g_{\mu\nu}$$

$$\frac{1}{2}\sqrt{-g} g^{\mu\nu} \partial g_{\mu\nu}$$

$$\delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu} \partial g_{\mu\nu} \tag{2.2}$$

An immediate consequence of equation (2.2) is

$$(\sqrt{-g})_{,\lambda} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu} g_{\mu\nu,\lambda} \tag{2.3}$$

## Example 2.3.1

Show that equation (2.3) leads to

$$\nabla_\mu(\sqrt{-g}) = 0$$

$\sqrt{-g}$ is a scalar density of weight 1. The covariant derivative of a scalar density of weight $a$ is

$$\nabla_\lambda \chi = \chi_{,\lambda} - \omega\, \Gamma^\mu_{\mu\lambda} \chi$$

We wish to show that

$$\nabla_\lambda(\sqrt{-g}) = (\sqrt{-g})_{,\lambda} - \sqrt{-g}\, \Gamma^\mu_{\mu\lambda} = 0$$

We know that

$$0 = \nabla_\mu\, g_{\nu\lambda} = g_{\nu\lambda,\mu} - g_{\nu\rho}\, \Gamma^\rho_{\lambda\mu} - g_{\lambda\rho}\, \Gamma^\rho_{\nu\mu}$$

$$\Rightarrow g_{\nu\lambda,\mu} = g_{\nu\rho}\, \Gamma^\rho_{\lambda\mu} + g_{\lambda\rho}\, \Gamma^\rho_{\nu\mu}$$

Equation (2.3) implies:

$$(\sqrt{-g})_{,\lambda} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}g_{\mu\nu,\lambda}$$

$$= \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}(g_{\mu\rho}\,\Gamma^{\rho}_{\nu\lambda} + g_{\nu\rho}\Gamma^{\rho}_{\mu\lambda})$$

$$= \frac{1}{2}\sqrt{-g}(\delta^{\nu}_{\rho}\,\Gamma^{\rho}_{\nu\lambda} + \delta^{\mu}_{\rho}\Gamma^{\rho}_{\mu\lambda})$$

$$= \frac{1}{2}\sqrt{-g}(\Gamma^{\nu}_{\nu\lambda} + \Gamma^{\mu}_{\mu\lambda})$$

$$= \sqrt{-g}\,\Gamma^{\mu}_{\mu\nu}$$

$$\Rightarrow \nabla_{\lambda}(\sqrt{-g}) = 0 \qquad \square$$

## 2.4 The Einstein Hilbert Action

We now consider the contribution to the action coming from the gravitational field:

$$S_g = \int_{\text{space}} \mathcal{L}_g d\Omega$$

The only scalar density of weight 1 involving the metric and its derivatives up to second order is $\sqrt{-g}R$. i.e. we take

$$\mathcal{L}_g = \kappa^{-1}\sqrt{-g}R = \kappa^{-1}\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu}$$

$$\Rightarrow \delta S_g = \kappa^{-1}\int [\delta(\sqrt{-g}\,g^{\mu\nu})R_{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu}]\delta\Omega$$

We require our expression for $\delta R_{\mu\nu}$ schematically, we have

$$R = \partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma$$

Thus

$$\delta R = \partial(\delta\Gamma) - \partial(\delta\Gamma) + \delta\Gamma\Gamma + \Gamma\delta\Gamma - \delta\Gamma\Gamma - \Gamma\delta\Gamma$$

$\delta\Gamma^{\mu}_{\nu\lambda}$ is a well defined tensor (even though $\Gamma^{\mu}_{\nu\lambda}$ is not, since this involves the difference of two connections) we can therefore convert partial derivatives to covariant derivatives:

$$\delta R^{\mu}_{\nu\lambda\rho} = (\delta\Gamma^{\mu}_{\nu\rho})_{;\lambda} - (\delta\Gamma^{\mu}_{\nu\lambda})_{;\rho}$$

The second term of the gravitational action is therefore

$$\kappa^{-1}\int\sqrt{-g}\left[(g^{\nu\rho}\delta\Gamma^{\mu}_{\nu\rho})_{;\mu}-(g^{\nu\rho}\delta\Gamma^{\mu}_{\nu\mu})_{;\rho}\right]d\Omega$$

Now $g^{\nu\rho}\delta\Gamma^{\mu}_{\nu\rho}$ and $g^{\nu\rho}\Gamma^{\mu}_{\nu\mu}$ are vectors, so we may apply the corollary to the divergence theorem to convert to a surface integral

$$\kappa^{-1}\int\sqrt{-g}(g^{\nu\mu}\delta\Gamma^{\rho}_{\nu\mu}-g^{\nu\rho}\delta\Gamma^{\mu}_{\nu\mu})d\Sigma_{\rho}=0)$$

The gravitational action reduces to

$$\delta S_g=\kappa^{-1}\int\delta(\sqrt{-g}\,g^{\mu\nu})R_{\mu\nu}d\Omega$$

$$=\kappa^{-1}\int[\delta(\sqrt{-g})g^{\mu\nu}R_{\mu\nu}+\sqrt{-g}\delta g^{\mu\nu}R_{\mu\nu}]d\Omega$$

$$=\kappa^{-1}\int\left(\frac{1}{2}\sqrt{-g}\,g^{\lambda\rho}\delta g_{\lambda\rho}g^{\mu\nu}R_{\mu\nu}-\sqrt{-g}\,g^{\mu\lambda}\delta g_{\lambda\rho}R_{\mu\nu}\right)d\Omega$$

$$=\kappa^{-1}\int\sqrt{-g}\left(\frac{1}{2}g^{\lambda\rho}R-R^{\lambda\rho}\right)\delta g_{\lambda\rho}d\Omega$$

$$=\kappa^{-1}\int\sqrt{-g}\,G^{\lambda\rho}\delta g_{\lambda\rho}d\Omega$$

## 2.5 Einstein's Field Equations

Putting the results together, we have

$$\delta S=\delta S_\mu+\delta S_g$$

$$=\int\frac{1}{2}\sqrt{-g}\,T^{\mu\nu}\delta g_{\mu\nu}d\Omega-\kappa^{-1}\int\sqrt{-g}\,G^{\mu\nu}\delta g_{\mu\nu}d\Omega$$

$$=\int\sqrt{-g}\left(\frac{1}{2}T^{\mu\nu}-\kappa^{-1}G^{\mu\nu}\right)\delta g_{\mu\nu}d\Omega=0$$

Since we assume metric variations vanish at the boundary, we must have

$$\frac{1}{2}T^{\mu\nu}-\kappa^{-1}G^{\mu\nu}=0$$

$$\implies G^{\mu\nu}=\frac{\kappa}{2}T^{\mu\nu}$$

In the weak field slow moving approximations, they reproduce Poisson's equation only when

$$\kappa = \frac{16\pi G}{c^4} \qquad \text{(D'Inverno)}$$

or, in natural units $c = g = 1, \kappa = 16\pi$.
$G^{\mu\nu} = \delta\pi T^{\mu\nu}$ Recall the twice contracted Bianchi identities

$$G^{\mu\nu}_{;\nu} = 0$$

$$\implies T^{\mu\nu}_{;\nu} = 0$$

## 2.6 Further Remarks on the Field Equations

The history of an isolated body in spacetime is a timelike world tube filled with the world lines of the constituent particles. Inside the world tube, we have $T^{\mu\nu} \neq 0$ , and we solve the non vacuum Einstein Field Equations.

$$G^{\mu\nu} = \delta\pi T^{\mu\nu}$$

Outside the world tube, $T^{\mu\nu} = 0$ and we solve the vacuum field equations

$$G^{\mu\nu} = 0 \iff R^{\mu\nu} = 0$$

Agreement with Newtonian limit requires $\kappa = 16\pi$.

$$\Rightarrow G = 8\pi T^{\mu\nu}$$

The world line of a particle $x^\mu(s)$ with non zero mass is timelike. Taking $s$ to be arc-length along the curve, we have

$$g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -1$$

If the particle is a test particle (doesn't perturb the geometry of spacetime), then the world line is a timelike geodesic satisfying

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds} = 0$$

where $s$ is now the proper time along the curve.

We take the world line of massless particles to be null geodesics

$$\frac{d^2x^\mu}{dr^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{dr}\frac{dx^\lambda}{dr} = 0$$

and

$$g_{\mu\nu}\frac{dx^\mu}{dr}\frac{dx^\nu}{dr} = 0$$

# 3 The Stress-Energy-Momentum Tensor

## 3.1 Decomposition of the Stress-Energy-Momentum Tensor in an Orthonormal Tetrad

The stress-energy-momentum tensor satisfies

$$T^{\mu\nu} = \frac{1}{8\pi} G^{\mu\nu} \quad \text{and} \quad T^{\mu\nu}_{;\nu} = 0$$

$T^{\mu\nu}$ is a symmetric 4x4 matrix and in general will have 4 mutually orthogonal eigenvectors; one timelike and three spacelike.
Let $u^\mu$ be the unit timelike eigenvector of $T^{\mu\nu}$ with eigenvalue $-\rho$, i.e.

$$\begin{aligned} T^{\mu\nu} u_\nu &= -\rho\, u^\mu, u^\mu u_\mu = 1 \\ &= -\rho\, G^{\mu\lambda} u_\lambda \end{aligned}$$

We take timelike worldlines tangent to $u^\mu$ (i.e. the integral curves of $u^\mu$) to be the worldlines of the constituent particles of the matter distribution. We take $\rho$ to be the proper density of the matter (density observed in the rest frame of the constituent particle). $u^\mu$ is the 4-velocity of a constituent particle, and it describes the interval motion of the body.

We further define

$$\begin{aligned} S_{\mu\nu} &= \rho u_\mu u_\nu - T_{\mu\nu} = S_{\mu\nu} \\ S_{\mu\nu} u^\nu &= \rho u_\mu (u_\nu u^\nu) - T_{\mu\nu} u^\nu \\ &= -\rho u_\mu + \rho u_\mu \\ &= 0 \end{aligned}$$

$u^\nu$ is a unit timelike eigenvector of $S_{\mu\nu}$ with eigenvalue zero.

$S_{\mu\nu}$ has 6 independent components, and is called the stress tensor of the matter distribution.

We now let $\{e^\mu_{(1)}, e^\mu_{(2)}, e^\mu_{(3)}\} = \{e^\mu_{(i)}\}^3_{i-1}$ be the unit spacelike eigenvectors of $T^{\mu\nu}$ with eigenvalues $\{p_{(i)}\}^3_{i=1}$, respectively.

$$T_{\mu\nu} e^\nu_{(i)} = p_{(i)} e_{(i)\mu} \quad i = 1, 2, 3 \quad \text{no sum over } i$$

Mutual orthogonality implies

$$u^\mu e_{(i)\mu} = u_\mu e^\mu_{(i)} = 0$$

Therefore

$$\begin{aligned} S_{\mu\nu} e^\nu_{(i)} &= \rho u_\mu u_\nu e^\nu_{(i)} - T_{\mu\nu} e^\nu_{(i)} \\ &= -\rho_{(i)} e_{(i)\mu} \quad i = 1, 2, 3 \end{aligned}$$

Hence $\{e^\nu_{(i)}\}^3_{i=1}$ are the unit spacelike eigenvectors of $S_{\mu\nu}$ with e-value $\{p_{(i)}\}^3_{i=1}$. These are called the 3 *principle stresses* in the matter distribution.
For pressures $p_{(i)} > 0$.
For tensions $p_{(i)} < 0$.

So we have 4 mutually orthogonal eigenvectors satisfying

$$\begin{aligned} u^\mu u_\mu &= -1 \\ e^\mu_{(i)} e_{(j)\mu} &= \delta_{(i)(j)} \\ e^\mu_{(i)} u_\mu &= 0 \end{aligned}$$

We set $u^\mu = e^\mu_{(0)}$, then we have

$$\begin{aligned} e^\mu_{(0)} e_{(0)\mu} &= -1 \\ e^\mu_{(1)} e_{(1)\mu} &= \delta_{(i)(j)} \\ e^\mu_{(0)} e_{(1)\mu} &= 0 \\ \Rightarrow e^\mu_{(a)} e_{(b)\mu} &= \eta_{(a)(b)} \qquad \text{(a,b=0,1,2,3)} \end{aligned}$$

(parenthesis around indices to distinguish tetrad indices from spacetime indices)

$$\begin{aligned} \Rightarrow e^\mu_{(a)} e_{(b)\mu} &= \eta(a)(B) \\ &= diag(-1, 1, 1, 1) \end{aligned}$$

$$\Rightarrow \boxed{g_{\mu\nu} e^\mu_{(0)} e^\nu_{(b)} = \eta_{(a)(b)}}$$

$\{e^\mu_{(a)}\}^3_{a=0}$ is an orthonormal tetrad. $\eta_{(a)(b)}$ are the components of the metric tensor on this orthonormal tetrad.

Any vector or tensor may be projected onto the tetrad from, for example, the components of the curvature tensor in the orthonormal tetrad are

$$R_{(a)(b)(c)(d)} = R\mu\nu\lambda\sigma e^\mu_{(a)} e^\nu_{(b)} e^\lambda_{(c)} e^\sigma_{(d)}$$

We can also write the metric components $g^{\mu\nu}$ in terms of $\{e^\mu_{(a)}\}$ following from the orthonormality conditions:

$$\boxed{g^{\mu\nu} = \eta^{(a)(b)} e^\mu_{(a)} e^\nu_{(b)}}$$

Therefore, we can pass freely from tensor components to tetrad components and vice-versa.

**Note:**

$$T_{(0)(0)} = T_{\mu\nu} e^{\mu}_{(0)} e^{\nu}_{(0)} = T_{\mu\nu} u^{\mu} u^{\nu} = -\rho u_{\nu} u^{\nu} = \rho$$

$$T_{(0)(i)} = T_{\mu\nu} e^{\mu}_{(0)} e^{\nu}_{(i)} = T_{\nu\mu} u^{\mu} e^{\nu}_{(i)} = 0 \qquad \text{(i=1,2,3)}$$

$$T_{(i)(j)} = T_{\mu\nu} e^{\mu}_{(i)} e^{\nu}_{(j)} = p_{(i)} e_{(i)\nu} e^{\nu}_{(j)} = p_{(i)} \delta_{(i)(j)}$$

$$\therefore T_{(a)(b)} = \text{diag}(\rho, p_{(1)}, p_{(2)}, p_{(3)})$$

$$g_{\mu\nu} e^{\mu}_{(a)} e^{\nu}_{(b)} = \eta_{(a)(b)} \qquad \text{(a)}$$

$$g^{\mu\nu} \eta^{(a)(b)} e^{\mu}_{(a)} e^{\nu}_{(b)} \qquad \text{(b)}$$

## 3.2 Stress-Energy-Momentum Tensor for a Perfect Fluid and for Dust

Writing (b) out explicitly:

$$\begin{aligned} g^{\mu\nu} &= \eta^{(0)(b)} e^{\mu}_{(0)} e^{\nu}_{(b)} + \eta^{(1)(b)} e^{\mu}_{(1)} e^{\nu}_{(b)} + \eta^{(2)(b)} e^{\mu}_{(2)} e^{\nu}_{(b)} + \eta^{(3)(b)} e^{\mu}_{(3)} e^{\nu}_{(b)} \\ &= \eta^{(0)(0)} e^{\mu}_{(0)} e^{\nu}_{(0)} + \eta^{(1)(1)} e^{\mu}_{(1)} e^{\nu}_{(1)} + \eta^{(2)(2)} e^{\mu}_{(2)} e^{\nu}_{(2)} + \eta^{(3)(3)} e^{\mu}_{(3)} e^{\nu}_{(3)} \\ &= u^{\mu} u^{\nu} + \sum_{i=1}^{3} e^{\mu}_{(i)} e^{\nu}_{(i)} \end{aligned}$$

$$\Rightarrow \boxed{\sum_{i=1}^{3} e^{\mu}_{(i)} e^{\nu}_{(i)} = g^{\mu\nu} + u^{\mu} u^{\nu}}$$

Recall

$$S_{\mu\nu} e^{\nu}_{(i)} = -p_{(i)} e_{(i)\mu}$$

$$\Rightarrow \sum_{i=1}^{3} S_{\mu\nu} e^{\nu}_{(i)} e^{\lambda}_{(i)} = -\sum_{i=1}^{3} p_{(i)} e_{(i)\mu} e^{\lambda}_{(i)}$$

$$\text{LHS} = S_{\mu\nu} \sum_{i=1}^{3} e^{\nu}_{(i)} e^{\lambda}_{(i)} = S_{\mu\nu}(g^{\nu\lambda} + u^{\nu} u^{\lambda}) = S^{\lambda}_{\mu} + 0$$

$$\Rightarrow \boxed{S^{\mu\nu} = \sum_{i=1}^{3} p_{(i)} e^{\mu}_{(i)} e^{\nu}_{(i)}}$$

For a perfect fluid, the stress is an isotropic pressure (no preferred direction)

$$p_{(1)} = p_{(2)} = p_{(3)} = p$$

$$\Rightarrow S_{\mu\nu} = -p\sum_{i=1}^{3} e^{\mu}_{(i)} e^{\nu}_{(i)}$$

$$\Rightarrow \boxed{S^{\mu\nu} = -p(g^{\mu\nu} + u^{\mu}u^{\nu})} \qquad \text{(stress tensor for a perfect fluid)}$$

where $4^{\mu}$ is the 4-velocity.

By definition, we have

$$\begin{aligned} T_{\mu\nu} &= \rho u_{\mu}u_{\nu} - S_{\mu\nu} \\ &= \rho u_{\mu}u_{\nu} + p(g_{\mu\nu} + u_{\mu}u_{\nu}) \\ T_{\mu\nu} &= (\rho + p)u_{\mu}u_{\nu} + pg_{\mu\nu} \end{aligned}$$

(stress-energy-momentum tensor for a perfect fluid)

## Example 3.2.1

Show that for incoherent matter with proper density $\rho$, that $\rho$ changes along integral curves of $u^{\mu}$ according to

$$\rho_{,\mu}u^{\mu} = \rho u^{\mu}_{;\mu} = 0$$

Further show that the world lines of the dust particles are timelike geodesics. We have $T^{\mu\nu} = \rho u^{\mu}u^{\mu}$. The conservation equations are

$$\begin{aligned} T^{\mu\nu}_{;\nu} &= 0 \\ \Rightarrow 0 &= \nabla_{\nu}(\rho u^{\mu}u^{\nu}) \\ &= \rho_{,\nu}u^{\mu}u^{\nu} + \rho(\nabla_{\nu}u^{\mu})u^{\nu} + \rho u^{\mu}\nabla_{\nu}u^{\nu} \\ &= u^{\mu}(\rho_{,\nu}u^{\nu} + \rho u^{\nu}_{;\nu}) + \rho u^{\mu}_{;\nu}u^{\nu} \qquad (1) \\ \Rightarrow 0 &= -(\rho_{,\nu}u^{\nu} + \rho u^{\nu}_{;\nu}) + \rho u_{\mu}u^{\mu}_{;\nu} \\ \text{But} \quad u_{\mu}u^{\mu}_{;\nu} &= \frac{1}{2}(u_{\mu}u^{\mu})_{;\nu} = 0 \qquad \text{(as required)} \end{aligned}$$

Sub this result back into (1)

$$\begin{aligned} &\Rightarrow u^{\mu}_{;\nu}u^{\nu} = 0 \\ &\iff D_{u}u = 0 \end{aligned}$$

i.e. the integral curve of the dust particle parallel transports its own tangent vector $\Rightarrow$ geodesics.

# 4 The Schwarzschild Solution

## 4.1 Canonical Form of a Spherically Symmetric Line-Element

We shall consider spherically symmetric solutions to Einstein's vacuum field equations.

Spherical symmetry implies that there exists a coordinate system $(t, r, \theta, \varphi)$ say, in which the line-element is invariant under the reflections

$$\theta \to \theta' = \pi - \theta$$
$$\varphi \to \varphi' = -\varphi$$

i.e. no cross terms of the form $drd\theta, drd\varphi, d\theta d\varphi, d\theta d\varphi, dtd\theta, dtd\varphi$ and that each 2D submanifold defined by $t =$const, $r =$const, are the 2-spheres.

$$dl^2 = a^2(d\theta^2 + sin^2\theta d\varphi^2)$$

Therefore, the spherically symmetric line-element has the form

$$ds^2 = -A(r,t)dt^2 + 2B(r,t)dtdr + C(t,r)dr^2 + D(t,r)(d\theta^2 + sin^2\theta d\varphi^2)$$

Changing the radial coordinate $r \to \tilde{r} = \sqrt{D}$

$$\Rightarrow ds^2 = -\tilde{A}(t,\tilde{r})dt^2 + 2\tilde{B}(t,\tilde{r})dtd\tilde{r} + \tilde{C}(t,\tilde{r})d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + sin^2\theta d\varphi^2)$$

Introduce a new time coordinate by

$$d\tilde{t} = I(t,\tilde{r})[-\tilde{A}(t,\tilde{r})dt + \tilde{B}(t,\tilde{r})d\tilde{r}]$$

$$\Rightarrow d\tilde{t}^2 = I(t,\tilde{r})[\tilde{A}^2dt^2 - 2\tilde{A}\tilde{B}dtd\tilde{r} + \tilde{B}^2d\tilde{r}^2]$$

$$\Rightarrow -\tilde{A}dt^2 + 2\tilde{B}dtd\tilde{r} = -\frac{d\tilde{t}^2}{I^2\tilde{A}} + \frac{\tilde{B}^2}{\tilde{A}}d\tilde{r}^2$$

The line-element now reads

$$ds^2 = -\frac{dt^2}{I^2A} + \left(\frac{B^2}{A} + C\right)dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2) \quad \text{(dropped the tildes)}$$

Defining 2 new functions $p = p(t,r); q = q(t,r)$ by

$$\frac{1}{I^2A} = e^p; \quad \frac{B^2}{A} + C = e^q$$

Our canonical form of a spherically symmetric line-element reads

$$ds^2 = -e^p dt^2 + e^q dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

## 4.2 The Schwarzschild Solution

To determine the functions $p(t,r), q(t,r)$ we must solve the vacuum field equations $G^{\mu}_{\nu} = 0$. The non vanishing components of the Einstein tensor are

$$G^t_t = -e^{-q}\left(\frac{1}{r}\frac{\partial q}{\partial r} - \frac{1}{r^2}\right) - \frac{1}{r^2} \tag{i}$$

$$G^2_t = \frac{e^{-q}}{r}\frac{\partial q}{\partial t} \tag{ii}$$

$$G^r_r = e^{-q}\left(\frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \tag{iii}$$

$$G^\theta_\theta = G^\varphi_\varphi = -\frac{1}{2}e^{-q}\left(\frac{1}{2}\frac{\partial p}{\partial r}\frac{\partial q}{\partial r} + \frac{1}{r}\frac{\partial q}{\partial r} - \frac{1}{r}\frac{\partial p}{\partial r} - \frac{1}{2}\left(\frac{\partial p}{\partial r}\right)^2 - \frac{\partial^2 p}{\partial r^2}\right)$$
$$-\frac{1}{2}e^{-p}\left(\frac{\partial^2 q}{\partial t^2} + \frac{1}{2}\left(\frac{\partial q}{\partial t}\right)^2 - \frac{1}{2}\frac{\partial 1}{\partial t}\frac{\partial p}{\partial t}\right)$$

We see that the Einstein equations give us 4 non trivial equations. However, they are not all independent. The twice contracted Bianchi identities $G^{\mu\nu}_{;\nu} = 0$ imply that vanishing of $(i) - (iii)$ implies vanishing of $(iv)$. So we have 3 independent equations

$$e^{-q}\left(\frac{1}{r}\frac{\partial q}{\partial r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0 \tag{a}$$

$$\frac{\partial q}{\partial t} = 0 \tag{b}$$

$$e^{-q}\left(\frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0 \tag{c}$$

It is immediately obvioius from $(b)$ that $q$ is a function of $r$ only. i.e $q = q(r)$ and therefore, $(a)$ becomes a simple ODE:

$$e^{-q} - e^{-q} r\frac{dq}{dr} = 1$$
$$\Rightarrow \frac{d}{dr}(re^{-q}) = 1$$
$$\Rightarrow re^{-q} = r = \text{ const}$$

Taking our constant of integration to be $-2M$ (which we will interpret later) yields:

$$e^{-q} = \left(1 - \frac{2M}{r}\right)^{-1}$$

To optain $p$ we note that adding $(a)$ and $(c)$ gives

$$\frac{\partial p}{\partial r} + \frac{\partial q}{\partial r} = 0$$
$$\text{i.e. } p + q = f(t)$$

$$\begin{aligned} \Rightarrow e^{p} &= e^{-q} e^{f(t)} \\ &= \left(1 - \frac{2M}{r}\right) e^{f(t)} \end{aligned}$$

Our line element reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right) e^{f(t)} dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + sin^2\theta d\varphi^2\right)$$

Finally, we may eliminate $f(t)$ by redefining our time coordinate by

$$e^{\frac{1}{2}f(t)} dt = dt'$$
$$\Rightarrow t' = \int_c^t e^{\frac{1}{2}f(u)} du$$

which gives (after dropping primes)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + sin^2\theta d\varphi^2)$$

(Schwarzschild Solution)

## 4.3 Properties of the Schwarzschild Solution

### 4.3.1 Limiting Cases $M \to 0, r \to \infty$

It is clear that by setting $M = 0$ we retrieve the Minkowski metric in spherical polar coordinates. The parameter $M$ represents the mass/energy and one may interpret the Schwarzschild solution as the geometry due to a point mass $M$ at the origin.

We further note that as $r \to \infty$, we again retrieve the Minkowski metric. We did not impose asymptotic flatness! Spherically symmetric vacuum solutions of Einstein's equations are necessarily asymptotically flat.

### 4.3.2 The Coordinate Singularity at $r = 2M$

The metric components of $G_{\mu\nu}$ are singular at $r = 0$, and $r = 2M$ ($r = \frac{2GM}{C^2}$ in non natural units). The $r = 0$ singularity is known as a curvature singularity and is irremovable. The $r = 2M$ singularity is a coordinate singularity and may be rmoved by an appropriate coordinate transformation (though $r = 2M$ still has important physical implications). To see this, we make the coordinate transformation $(t, r, \theta, \varphi) \to (u, r, \theta, \varphi)$ where

$$u = t - r - 2Mlog(r - 2M)$$
$$\Rightarrow du = dt - \left(1 - \frac{2M}{r}\right)^{-1} dr$$

In these coordinates, the metric reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right)^{-1} du^2 - 2dudr + r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

In coordinates $(u, r, \theta, \varphi)$ the components of $G_{\mu\nu}$ are non singular at $r = 2M$

$$g_{\mu\nu} = \begin{pmatrix} -(1 - \frac{2M}{r}) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix}$$

We also note that in the standard form of the Schwarzschild metric

$$g_{tt} = -\left(1 - \frac{2M}{r}\right) < 0, \quad r > 2M$$
$$> 0, \quad r > 2M$$

so that the signature of the metric is $(+ - ++)$ for $r < 2M$. In this region, $r$ takes on the character of a time coordinate and $t$ a spatial coordinate. We call the region $r > 2M$ the exterior Schwarzschild geometry, and the region $0 < r < 2M$ the interior Schwarzschild geometry.

## 4.4 Birkhoff's Theorem

**Definition:** Static space time: A space time is said to be static if there exists a coordinate system in which the metric components are time- independent and the metric is time reversal invariant, i.e. there exists a coordinate system such that $g_{\mu\nu,t} = 0$, and there are no cross terms $dtdx^i \quad (i = 1, 2, 3)$

**Note:** The chart independent definition relies on the existence of a time-like killing vector that is hypersurface orthogonal.

We note that the Schwarzschild solution is static, but we did not impose this!

**Birkhoff's Theorem:** A spherically symmetric vacuum solution in the exterior region is necessarily static.

**Corollary:** For a spherically symmetric source in the region $r \leq a$, where $a > 2M$, the exterior Schwarzschild solution is the unique solution.

# 5 Solar System Tests of GR

In order for GR to be considered a viable theory of gravitation, it ought to be able to explain various phenomena in our solar system such as light deflection

We model the gravitational field by the Schwarzschild solution with $M = M_\odot$, the mass of the sun. We model the planets as text particles which move on timelike geodesics of the Schwarzschild spcacetime. There are 3 classical tests we shall consider:

## 5.1 The Gravitational Red-Shift

Let $C_o$ and $C_1$ be the timelike world lines of an emitter and receiver of light, respectively. Let $\tau$ be the proper time along them. Let $P_0$ and $P_1$ be the null worldline of a photon emitted at the even $P_0$ on $C_0$ and received at the event $P_1$ on $C_1$. Suppose in a short interval $d\tau_0$ of proper time on $C_0$, $n$ photons are emitted and these are received in an interval $d\tau_1$ of proper time on $C_1$. Then,

$$\begin{aligned}
\nu_0 &= \text{frequency of emission} \\
&= \text{no. of photons per unit time} \\
&= \frac{n}{d\tau_0}
\end{aligned}$$

Similarly,

$$\begin{aligned} \nu_1 &= \text{frequency of reception} \\ &= \frac{n}{d\tau_1} \\ \Rightarrow \nu_0 d\tau_0 &= \nu_1 d\tau_1 \\ \Rightarrow \frac{\nu_0}{\nu_1} &= \frac{d\tau_1}{d\tau_0} \end{aligned}$$

If $\lambda_0, \lambda_1$ are the emitted and received wavelengths respectively, then

$$\lambda_0 = \frac{1}{\nu_0} \qquad \lambda_1 = \frac{1}{\nu_1} \qquad \text{(c=1)}$$

$$\Rightarrow \frac{\lambda_1}{\lambda_0} = \frac{d\tau_1}{d\tau_0}$$

A signal is red shifted (loses energy) if $\lambda_1 > \lambda_0$ or if $d\tau_1 > d\tau_0$.

Suppose the emitter is at rest on the surface of the sun. Then the world line $C_0$ would be given by

$$\begin{aligned} r &= a = \text{solar radius} \\ \theta &= \theta_0 \\ \varphi &= \varphi_0 \end{aligned}$$

On $C_0$:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{a}\right) dt^2 \\ d\tau &= \sqrt{-ds^2} = \sqrt{1 - \frac{2M}{a}} dt \end{aligned}$$

Similarly, on $C_1$:

$$\begin{aligned} d\tau_1 &= \sqrt{1 - \frac{2M}{r}} dt \\ \Rightarrow \frac{\lambda_1}{\lambda_0} &= \frac{d\tau_1}{d\tau_0} = \frac{\sqrt{1 - \frac{2M}{r}}}{\sqrt{1 - \frac{2M}{a}}} \end{aligned}$$

For $\frac{M}{a}$ small $\left(\Rightarrow \frac{M}{r} \text{ small}\right)$

$$\begin{aligned} \frac{\sqrt{1 - \frac{2M}{r}}}{\sqrt{1 - \frac{2M}{a}}} &= \left(1 - \frac{M}{r} + O\left(\frac{M}{r}\right)^2\right)\left(1 + \frac{M}{a} + O\left(\frac{M}{a}\right)^2\right) \\ &\approx 1 + \frac{M}{a} - \frac{M}{r} \end{aligned}$$

$$\Rightarrow \boxed{\frac{\lambda_1}{\lambda_0} \approx 1 + \frac{M}{a} - \frac{M}{r}}$$

Since $\frac{M}{a} > \frac{M}{r}$, we have $\lambda_1 > \lambda_0$. i.e. signals are red-shifted as they pass through the gravitational field

$$z = \frac{\Delta\lambda}{\lambda_0} = \frac{\lambda_0 - \lambda_1}{\lambda_0} = 1 - \frac{\lambda_1}{\lambda_0} = -\left(\frac{M}{a} - \frac{M}{r}\right)$$

(or $-\frac{GM}{c^2}\left(\frac{1}{a} - \frac{1}{r}\right)$ in standard units)

**Note:** This is not a Doppler shift since there is no relative motion between observers.

## 5.2 Planetary Motion and Perihelian Advance of Mercury

### 5.2.1 Geodesic Equations

We treat planets as test particles moving among timelike geodesics of Schwarzschild spacetime. Line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) = -1 \tag{5.1}$$

$$\dot{t} = \frac{dt}{d(\text{proper time})}$$

$$E - L = \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0$$

$$
\begin{aligned}
\boldsymbol{\mu = t :} & \frac{d}{d\tau}\left[-2\left(1-\frac{2M}{r}\right)^{-1}\dot{r}\right] \\
& \Rightarrow \left(1-\frac{2M}{r}\right)\dot{t} = E \\
\boldsymbol{\mu = r :} & 0 = \frac{d}{d\tau}\left[\left(1-\frac{2M}{r}\right)^{-1}\dot{r}\right] - \left[-\frac{2M}{r^2}\dot{t}^2 - \left(1-\frac{2M}{r}\right)^{-2}\frac{2M}{r^2}\dot{r}^2 + 2r(\dot{\theta}^2+\sin^2\theta\dot{\varphi}^2)\right] \\
& \Rightarrow \ddot{r} = \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 - (2-2M)(\dot{\theta}^2+\sin^2\theta\dot{\varphi}^2) + \frac{M}{r^2}\left(1-\frac{2M}{r}\right)\dot{t}^2 = 0 \\
\boldsymbol{\mu = \theta :} & 0 = \frac{d}{d\tau}(2r^2\dot{\theta}) - 2r^2\sin\theta\cos\theta\dot{\varphi}^2 \\
& \Rightarrow \ddot{\theta} + \frac{2}{r}\dot{r}\theta^2 - \sin\theta\cos\theta\dot{\varphi}^2 \quad \text{(iv)} \\
\boldsymbol{\mu = \varphi :} & 0 = \frac{d}{d\tau}(2r^2\sin^2\theta\dot{\varphi}) \\
& \Rightarrow r^2\sin^2\theta\dot{\varphi} = h
\end{aligned}
$$

### 5.2.2 Propagation Equation for $\theta(\tau)$

(iv) $\Rightarrow \frac{\pi}{2}$ is a solution. Assume $\theta(0) = \frac{\pi}{2}, \dot{\theta}(0) = 0 \Rightarrow \ddot{\theta}(0) = 0$.

Differentiating (iv) gives $\dot{\theta}(0) = 0$.
$\Rightarrow$ all derivatives of $\theta$ vanish.
Consider $\tau = \tau_1 > 0$ close to $\tau = 0$, then

$$
\begin{aligned}
\theta(\tau_0) &= \theta(0) + \dot{\theta}(0)\tau_1 + \frac{1}{2}\ddot{\theta}(0)\tau_1^2 + \frac{1}{3!}\dddot{\theta}(0)\tau_1^3 + ... \\
\dot{\theta}(\tau_1) &= \dot{\theta}(0) + \ddot{\theta}(0)\tau_1 + \frac{1}{2}\dddot{\theta}(0)\tau_1^2 + ... \\
\Rightarrow \theta(\tau_1) &= \frac{\pi}{2}, \dot{\theta}(\tau_1) = 0
\end{aligned}
$$

Therefore we have shown that assuming $\theta(0) = \frac{\pi}{2}, \dot{\theta}(0) = 0$, then it remains true for some nearby point. By induction, it is true for all values of $\tau$, $\theta(\tau) = \frac{\pi}{2}, \dot{\theta}(\tau) = 0$.
$\Rightarrow$ only consider equatorial plane. We now have

$$
\left(1-\frac{2M}{r}\right)\dot{t} = E
$$

$$r^2\dot{\varphi} = h$$
$$\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2M}{r}\right) - E^2 + \left(1 - \frac{2M}{r}\right) = 0$$

Substitution: $h = \frac{r_0}{r}$

$$\begin{aligned}\dot{r} &= \frac{dr}{d\tau} = \frac{dr}{du}\frac{du}{d\tau} \\ &= \frac{dr}{du}\frac{du}{d\varphi}\frac{d\varphi}{d\tau} \\ &= \frac{du}{d\varphi}\left(-\frac{r^2}{r_0}\right)\dot{\varphi} \\ &= -\frac{h}{r_0}\frac{du}{d\varphi}\end{aligned}$$

$$\frac{u^2}{r_0^2}\left(\frac{du}{d\varphi}\right)^2 + \frac{h^2}{r_0^2}u^2\left(1 - \frac{2M}{r_0}\right) - E^2 + 1 - \frac{2Mu}{r_0} = 0$$
$$\Rightarrow \left(\frac{du}{d\varphi}\right)^2 + u^2 = \frac{r_0^2}{h}(E^2 - 1) + \frac{2Mur_0}{h^2} + \frac{2Mu^3}{r_0}$$

(omitting $\frac{2Mu^3}{r_0}$, we retrieve the Newtonian result)

Differentiation gives the more familiar form

$$\frac{d^2u}{d\varphi^2} + u = \frac{Mr_0}{h^2} + \frac{3Mu^2}{r_0} \qquad \text{(Relativistic Binet Equation)}$$

### 5.2.3 Newtonian Result

Ignoring $\frac{2Mu^3}{r_0}$ and writing $\frac{2M}{r_0} = \varepsilon \ll 1$

$$\Rightarrow \left(\frac{du_N}{d\varphi}\right)^2 + u_N^2 = \frac{r_0^2}{h}(E^2 - 1) + \varepsilon\frac{u_N r_0^2}{h^2}$$

This can be solved exactly by writing the solution as $u_N + u_0 + v.$, where $u_0$ is a constant chosen to eliminate the term linear in $v$.

$$\Rightarrow \left(\frac{dv}{d\varphi}\right)^2 + u_0^2 + 2u_0v + v^2 = \frac{r_0^2}{h^2}(E^2 - 1) + \frac{\varepsilon r_0^2}{h^2}(u_0 + v)$$

$u_0$ is chosen such that

$$2u_0 = \frac{\varepsilon r_0^2}{h^2} \Rightarrow u_0 = \frac{1}{2}\frac{\varepsilon r_0^2}{h^2}$$

$$\Rightarrow \left(\frac{dv}{d\varphi}\right)^2 + v^2 = \frac{r_0^2}{h^2}(E^2-1) - u_0^2 + \frac{\varepsilon r_0^2 u_0}{h^2} = k^2$$

$$\Rightarrow v(\varphi) = k\sin(\varphi - \varphi_0)$$

$$\Rightarrow u_N = u_0(1 + e\sin(\varphi - \varphi_0))$$

with $e = \frac{k}{u}$ (defines ellipse for $0 < e < 1$)

$\left(\frac{du}{d\varphi}\right)^2 + u^2 = \frac{r_0^2(c^2-1)}{h^2} + 2\frac{Mur_0}{h^2} + 2\frac{Mu^3}{r_0}$

Newtonian result obtained by ignoring $u^3$ term. Solved with Ansatz

$$\begin{aligned} u_N &= u_0 + v \\ &= u_0(1 + esin(\varphi - \varphi_0)) \end{aligned}$$

ellipse with period $2\pi$

### 5.2.4 Shape of General Relativistic Orbit

Again we take $u = u_0 + v$ where $u_0$ is a constant chosen to eliminate the term linear in $v$.
⇒ This leads to requiring that $u_0$ satisfies the quadratic

$$3\epsilon u_0^2 - 2u_0 + \epsilon\frac{r_0^2}{h^2} = 0$$

where $\epsilon = \frac{2M}{r_0} \ll 1$ and we choose the solution that is closest to the Newtonian result. Then $v$ satisfies

$$\left(\frac{dv}{d\varphi}\right)^2 + u_0^2 + v^2 = r_0^2\frac{(c^2-1)}{h^2} + 2\frac{Mr_0u_0}{h^2} + \epsilon u_0^3 + 3\epsilon u_0 v^2 + \epsilon v^3$$

Ignoring the $v^3$ term and collecting constants

$$\left(\frac{dv}{d\varphi}\right)^2 + v^2(1 - 3\epsilon u_0) = k^2$$

which is easily solved, yielding

$$v = \frac{k}{\omega} sin\omega(\varphi - \varphi_0)$$

where $\omega^2 = 1 - 3\epsilon u_0$. i.e. the shape of the orbit as predicted by relativity is an ellipse with a periodicity

$$\frac{2\pi}{\omega} \approx 2\pi(1 + \frac{3}{2}\epsilon u_0)$$

the periolian advance is given by (in standard units)

$$\begin{aligned}\Delta\varphi &= 3\pi\epsilon u_0 \\ &= 6\pi\frac{GM}{c^2}\left(\frac{u_0}{r_0}\right)\end{aligned}$$

To approximate $\frac{u_0}{r_0}$ we use the fact that each orbit is approximately Newtonian and we know for an ellipse

$$r_{\text{max}} = a(1+e) \quad r_{\text{min}} = a(1-e)$$

where $a$ is the semi-major axis

$$\begin{aligned}(u_N)_{\text{max}} &= u_0(1+e) = \frac{r_0}{r_{\text{min}}} = \frac{r_0}{a(1-e)} \\ (u_N)_{\text{min}} &= u_0(1-e) = \frac{r_0}{r_{\text{max}}} = \frac{r_0}{a(1+e)} \\ \Rightarrow \frac{2u_0}{r_0} &= \frac{1}{a(1-e^2)} + \frac{1}{a(1-e^2)} = \frac{2}{a(1-e^2)}\end{aligned}$$

$$\therefore \boxed{\Delta\varphi = \frac{6\pi GM}{c^2 a(1-e^2)}}$$

For Mercury, this predicts a shift of 43" per century while the observed value is 43".1 $\pm$ 0.5

## 5.3 Light Reflection

We consider photon paths in the Schwarzschild gravitational field. We describe the photons by null geodesics

$$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\lambda}\dot{x}^\mu\dot{x}^\lambda = 0$$

and

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

where $\dot{x}^\mu = \frac{dx^\mu}{ds}$ is an affine parameter.
Again, without loss of generality, we take the photon path to be in the equatorial plane $\theta(s) = \frac{\pi}{2}$ for all $s$. Our geodesic equations are

$$\left(1 - \frac{2M}{r}\right)\dot{t} = E$$

$$r^2\dot{\varphi} = h$$

$$\ddot{r} - \frac{M^2}{r^2}\left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 - (r - 2M)\dot{\varphi}^2 + \frac{M}{r}\left(1 - \frac{2M}{r}\right)\dot{t}^2 = 0$$

The 1st integral of the motion

$$\begin{aligned} 0 &= g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \\ &= -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + r^2\dot{\varphi}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 \end{aligned}$$

Using the fact that

$$\dot{r} = \frac{dr}{d\varphi}\dot{\varphi} = \frac{h}{r^2}\frac{dr}{d\varphi}$$

and the conservation equations to simplify

$$\left(1 - \frac{2M}{r}\right)^{-1}\frac{h^2}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{h^2}{r^2} - E^2\left(1 - \frac{2M}{r}\right)^{-1} = 0$$

Again, we take $u = \frac{r_0}{r}$

$$\Rightarrow \left(\frac{du}{d\varphi}\right)^2 + u^2 - \frac{2Mu^3}{r_0} = \frac{E^2 r_0^2}{h^2}$$

or

$$\boxed{\left(\frac{du}{d\varphi}\right)^2 + u^2 - \epsilon u^3 = \frac{E^2 r_0^2}{h^2}}$$

where $\epsilon = \frac{2M}{r_0} \ll 1$
Take $u = u_0 + \epsilon u$, and subbing into our equation and equating equal orders of $\epsilon$ gives

$$\left(\frac{du_0}{d\varphi}\right)^2 + u_0^2 = \frac{r_0^2}{d^2} \quad \text{where } d = \frac{h}{E} \tag{A}$$

and

$$2\left(\frac{du_0}{d\varphi}\right)\left(\frac{du_1}{d\varphi}\right) + 2u_0u_1 - u_0^3 = 0 \tag{B}$$

Equation (A) is easily solved

$$u_0 = \frac{r_0}{d}\sin\varphi \qquad \text{taking} \quad \varphi_0 = 0$$

Then subbing this into equation (B)

$$\cos\varphi\left(\frac{du_1}{d\varphi}\right) + \sin\varphi u_1 - \frac{1}{2}\frac{r_0^2}{d^2}\sin^3\varphi = 0$$

Try a solution of the form

$$u_1 = A + B\sin\varphi + C\cos^2\varphi$$

$$\Rightarrow B + \sin\varphi(A - C) + \sin^3\varphi(C - \frac{1}{2}\frac{r_0^2}{d^2}) = 0$$

$$\Rightarrow B + 0, A = C, C = \frac{1}{2}\frac{r_0^2}{d^2}$$

$$\Rightarrow u_1 = \frac{1}{2}\frac{r_0^2}{d^2}(1 + \cos^2\varphi)$$

$$\Rightarrow u = \frac{r_0}{d}\sin\varphi + \frac{\epsilon}{2}(a + \cos^2\varphi)$$

We require the total deflection in the asymptotic regions $r \to \infty (u \to 0)$.
$r \to \infty$, as $\varphi \to -\varphi_1$
$r \to \infty$, as $\varphi \to \pi + \varphi_2$
subbing these into our equation gives

$$0 = \frac{r_0}{d}(-\varphi_1) + \epsilon\frac{1}{2}\frac{r_0^2}{d^2}(1 + 1 + O(\epsilon^2))$$

$$\Rightarrow \varphi_1 = \frac{r_0}{d}\epsilon$$

$$0 = -\frac{r_0}{d}\varphi_2 + \epsilon\frac{1}{r}\frac{r_0^2}{d^2}(1 + 1 + O(\epsilon^2))$$

$$\Rightarrow \varphi_2 = \frac{r_0}{d}\epsilon$$

$$\therefore \Delta\varphi = \varphi_1\varphi_2 = \frac{2r_0 t}{d} = \frac{4M}{d}$$

$$\boxed{\Delta\varphi = \frac{4GM}{c^2 d}} \qquad \text{(total deflection angle (in standard units))}$$

Take $M = M_\odot, d = R_\odot$, gives $\Delta\varphi = 1".75$. Observed in 1919 by Sir Arthur Eddington during a solar eclipse.

# 6 Black Holes

## 6.1 Radial In-falling Photons

Consider an observer at rest relative to the source of the Schwarzschild gravitational field. The observer's world line is $r = \text{constant}, \theta = \text{constant}, \varphi = \text{constant}$ and

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2$$

where $\tau$ is proper time

$$\therefore \frac{d\tau}{dt} = \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}$$

For $r \gg 2M$, then along the observer is world line

$$\frac{d\tau}{dt} = 1 \qquad \Rightarrow t = \tau (\text{choosing } \tau(0) = 0)$$

Therefore, $t$ corresponds to proper time measured by an observer at rest at infinity. How does such an observer 'see' a radially in-falling photon as $r \to 2M$?

The world line of a radially in-falling photon satisfies

$$\left(1 - \frac{2M}{r}\right)^{-1} dr^2 = \left(1 - \frac{2M}{r}\right) dt^2$$

$$\Rightarrow \boxed{\frac{dt}{dr} = \pm \left(1 - \frac{2M}{r}\right)^{-1}}$$

where + represents an outgoing photon and - represents an ingoing photon.

Solving gives

$$\begin{aligned} t &= \pm(r + 2M \log(r - 2M) + C) \\ \Rightarrow u &= t \mp (r + 2M \log(r - 2M)) \\ &= \text{constant along radially null geodisics.} \end{aligned}$$

Clearly, as $r \to 2M, t \to \infty$. i.e. an observer at infinity will never 'see' the photon cross the horizon ($r = 2M$), according to this observer, it takes an infinite amount of time to reach $r = 2M$.

**Note:** As $r \to \infty$, we have $\frac{dt}{dr} = \pm 1 \Rightarrow t = \pm r + c$.
i.e. as $r \to \infty$, ingoing and outgoing null rays are straight lines with angle $\pm 45°$.

## 6.2 Radially In-falling Particles

A radially in-falling particle will move on a timelike geodesic given by

$$\left(1-\frac{2M}{r}\right)\dot{t}=E$$

$$-\left(1-\frac{2M}{r}\right)\dot{t}^2=\left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2=-1$$

If we consider a particle initially at rest at infinity

$$\Rightarrow E=1$$

Then the geodesic equations give

$$-\left(1-\frac{2M}{r}\right)^{-1}+\left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2=-1$$

$$\Rightarrow \left(\frac{d\tau}{dr}\right)^2=\frac{r}{2M}$$

$$\Rightarrow \frac{d\tau}{dr}=-\left(\frac{r}{2M}\right)^{\frac{1}{2}}$$

(minus sign reflects the fact that the particle is ingoing.
Integrating, we obtain

$$\tau=\frac{2}{3(2M)^{\frac{1}{2}}}(r_0^{\frac{3}{2}}-r^{\frac{3}{2}})$$

where the particle is at $r_0$ at $\tau=\tau_0$. Now the proper time to reach the singularity $r=0$ is

$$\tau=\tau_0+\frac{2}{3(2M)^{\frac{1}{2}}}r_0^{\frac{3}{2}}$$

which is finite.

According to his clock, he passes through the coordinate singularity $r=2M$ continuously, and reaches the curvature singularity $r=0$ in a finite proper time.

If we now describe the motion in terms of coordinate time $t$ (time measured by an observer at rest at infinity), then

$$\frac{dt}{dr}=\frac{\dot{t}}{\dot{r}}=\sqrt{\frac{r}{2M}}\left(1-\frac{2M}{r}\right) \qquad \text{(E=1)}$$

Integrating, we obtain

$$t - t_0 = -\frac{2}{3(2M)^{\frac{1}{2}}}(r^{\frac{3}{2}} - r_0^{\frac{3}{2}} + 6Mr^{-\frac{1}{2}} - 6Mr_0^{\frac{1}{2}})$$
$$+ 2M \log\left(\frac{(r^{\frac{1}{2}} + (2M)^{\frac{1}{2}})(r_0^{\frac{1}{2}} - (2M)^{\frac{1}{2}})}{(r_0^{\frac{1}{2}} + (2M)^{\frac{1}{2}})(r^{\frac{1}{2}} - (2M)^{\frac{1}{2}})}\right)$$

$t \to \infty$, as $t \to 2M$. So again, an observer at infinity never 'sees' the particle cross the horizon despite the fact that according to a clock attached to the particle, it reaches the singularity in an infinite time. According to his clock, he passes through the coordinate singularity $r = 2M$ continuously, and reaches the curvature singularity $r = 0$ in a finite proper time.

The path of a photon is always tangent to the local radial null cone and here two photon paths passing through $P$ define the local null. The paths of massive particles are always inside the null cone. In there $(t, r)$ coordinates, the local radial null cones are closing as $r \to 2M$. For $r < 2M$, the null cones tip over and we can see that an observer cannot remain at rest but is forced to move towards the singularity.

## 6.3 The Kruskal Extension of the Schwarzschild Manifold

It is clear that the coordinates $(t, r)$ are a bad choice for wollowing an infalling particle.

We look for new coordinates $(u, v)$ in terms of which the local radial null cones do not close as $r \to 2M$. We employ Kruskal coordinates:

$$u = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)$$

$$v = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right)$$

$$du = \frac{r}{8M^2}\left(\frac{r}{2M} - 1\right)^{-\frac{1}{2}} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right) dr + \frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right) dt$$

$$dv = \frac{r}{8M^2}\left(\frac{r}{2M} - 1\right)^{-\frac{1}{2}} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right) dr + \frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right) dt$$

$$\Rightarrow du^2 - dv^2 = \frac{r}{32M^3} e^{\frac{r}{2M}} \left[ -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 \right]$$

$$\Rightarrow ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} (du^2 - dv^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

where $r = r(u, v)$ is defined implicitly by

$$u^2 - v^2 = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}}$$

**Remarks:**

1. The line element is singular only at $r = 0$.

2. $r = 0 \iff u^2 - v^2 = -1$ hyperbola with $u, v$ as Cartesian coordinates.

   $r = \text{constant} > 2M \iff u^2 - v^2 = \text{positive constant}$
   $r = \text{constant} < 2M \iff u^2 - v^2 = \text{negative constant}$

3. Null radial geodesics are now defined by $\frac{dv}{du} = \pm 1$. i.e. in $\mathbb{R}(u, v)$ are Cartesian coordinates with local null cones as straight lines at 45°. This is because $(u, v)$ was chosen to satisfy

$$-\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 = F(u, v)(du^2 - dv^2)$$

4. 

$$\tanh\left(\frac{t}{4M}\right) = \frac{v}{u}.$$

   $t = \text{constant} \Rightarrow v = \text{constant}.u$

5. $r = 2M \iff u = \pm v$.

**Conclusions from Space-Time Diagram in Kruskal Coordinates**

- $r = 2M$ is a null-cone (null hypersurface) separating I and III from II and IV respectively.

- Massive particles and photons can cross the radius $r = 2M$ from I to II but *not* from II to I. Hence, $r = 2M$ is a 'one-way membrane' known as the event horizon; it is the boundary of the Schwarzschild black hole.

- Region I corresponds to the exterior Schwarzschild solution ($r > 2M$), describing the gravitational field outside a spherically symmetric object of mass $M$.

- Region II represents a black hole solution. Observers inside this region cannot send signals to an observer in region I and all observers in this region are destined for the future singularity $r = 0$.
- Region III is a universe whose geometry is identical to that of Region I, though the two regions are causally disconnected, i.e. no communication either way is possible.
- Region IV is the Schwarzschild white hole, the time reversal of a black hole. Generally not thought to be physical since "nature abhors naked singularities."

# 7 Cosmology

## 7.1 The Cosmological Principle:

Our position in the univers, with respect to the largest scales, is in no sense preferred. Extends the familiar Copernician Principle which states that our position in the solar system is in no sense preferred.

The cosmological principle is modelled by asserting that the universe is globally spatially isotropic. (globally isotropy implies homogeneity.)

**Definition** Homogeneous: For each space-like hypersurface, there are no privileged points, i.e. each point is a centre of spherical symmetry.

**Definition** Isotropy: For each space-like hyperspace, there are no privileged directions about any point.

## 7.2 Kinematics of the Continuum

### 7.2.1 Connecting Vectors

We consider the spacetime of the universe to be filled with the trajectories of massive particles (the galaxies) forming a congruence of timelike world lines with one world line passing through each point of the space time.

Let $\Sigma$ be a space like hypersurface, then the normal to $\Sigma$ is timelike and vectors tangent to $\Sigma$ are spacelike.

Let $\{\xi^i\}_{i=3}^{3}$ be an intrinsic coordinate system on $\Sigma$. Since there is a line of the congruence passing through every point of $\Sigma$, we can use the points of $\Sigma$ to label the lines of the congruence.

The congruence is timelike, so we use proper time $\tau$ as parameter along each worldline. Then for $\{xi^\mu\}_{\mu=0}^3$ a chart on the manifold, the parametric equations of the congruence are

$$x^\mu = x^\mu(\tau, \xi^i); \quad \mu = 0, ..., 3; \quad i = 1, ..., 3.$$

A particular line of the congruence is given by

$$\xi^i = \text{constants}; \quad x^\mu = x^\mu(\tau, \xi^i),$$

the unit tangent to this line is

$$u^\mu = \frac{\partial x^\mu}{\partial \tau}; \quad u^\mu u_\mu = -1$$

the 4-velocity of particle with worldline $\xi^i$.

Consider now two neighbouring lines of the congruence $\xi^i$ and $\xi^i + \delta\xi^i$. $\zeta^\mu$ is an infinitesimal connecting vector defined along $\xi^i$.

$\zeta^\mu$ connects points of equal parameter value $\tau$ on $\xi^i$ and $\xi^i + \delta\xi^i$

$$\Rightarrow \zeta^\mu = x^\mu(\tau, \xi^i + \delta\xi^i) - x^\mu(\tau, \xi^i)$$

Taylor expanding for small $\delta\xi^j$.

$$\boxed{\zeta^\mu = \frac{\partial x^\mu(\tau, \xi^i)}{\partial \xi^j} \delta\xi^j}$$

So see how $\zeta^\mu$ varies along the line of congruence $\xi^i$, we differentiate with respect to $\tau$

$$\begin{aligned} \frac{\partial \zeta^\mu}{\partial \tau} &= \frac{\partial}{\partial t} \left( \frac{\partial x^\mu(\tau, \xi^i)}{\partial \xi^j} \right) \delta\xi^j \\ &= \frac{\partial}{\partial \xi^j} \left( \frac{\partial x^\mu(\tau, \xi^j)}{\partial \tau} \right) \delta\xi^j \\ &= \frac{\partial u^\mu}{\partial \xi^j} \delta\xi^j \end{aligned}$$

But

$$\begin{aligned}
\frac{\partial u^\mu}{\partial \xi^j} &= \frac{\partial u^\mu}{\partial x^\nu}\frac{\partial x^\nu}{\partial \xi^j} \\
\Rightarrow \frac{\partial \zeta^\mu}{\partial \tau} &= u^\mu{}_{,\nu}\frac{\partial x^\nu}{\partial \xi^j}\delta\xi^j \\
&= u^\mu{}_{,\nu}\zeta^\nu \\
\iff \frac{\partial \zeta^\mu}{\partial x^\nu}\frac{\partial x^\nu}{\partial \tau} &= u^\mu{}_{,\nu}\zeta^\nu \\
\iff \zeta^\mu{}_{,\nu}u^\nu &= u^\mu{}_{,\nu}\zeta^\nu \\
\iff [\zeta, u]^\mu &= 0
\end{aligned}$$

We also know that

$$X^\mu{}_{,\nu}Y^\nu - Y^\mu{}_{,\nu}X^\nu = X^\mu{}_{;\nu}Y^\nu - Y^\mu{}_{;\nu}X^\nu$$

$\therefore$ The propagation equation may be rewritten

$$\zeta^\mu{}_{;\nu}u^\nu = u^\mu{}_{;\nu}\zeta^\nu$$

$$\iff \boxed{\dot{\zeta}^\mu = u^\mu{}_{;\nu}\zeta^\nu}$$

where

$$\dot{\zeta}^\mu = \frac{D\zeta^\mu}{d\tau} = \zeta^\mu{}_{;\nu}u^\nu$$

We may also define the orthogonal connecting vector

$$\eta^\mu = h^\mu{}_\nu\zeta^\nu$$

where $h^\mu{}_\nu = \delta^\mu_\nu + u^\mu u_\nu$ is the projection tensor which projects vectors orthogonal to $u^\mu$.

$$\begin{aligned}
h^\mu{}_\nu &= \delta^\mu_\nu + u^\mu u_\nu \\
\iff h^{\mu\nu} &= g^{\mu\nu} + u^\mu u^\nu \\
\iff h_{\mu\nu} &= g_{\mu\nu} + u_\mu u_\nu
\end{aligned}$$

It is straight forward to verify that $h^\mu{}_\nu$ satisfies

i. $h^\mu{}_\nu u^\nu = 0$

ii. $h^\mu{}_\nu h^\nu{}_\lambda = h^\mu{}_\lambda$

iii. $h^{\mu}{}_{\mu} = 3$

Arbitrary tensors may be projected onto the plane orthogonal to $u^{\mu}$, e.g. for a $\binom{2}{0}$ tensor $Q^{\mu\nu}$, its projection orthogonal to $u^{\mu}$ is

$$\tilde{Q}^{\mu\nu} = h^{\mu}{}_{\lambda} h^{\nu}{}_{\rho} Q^{\lambda\rho}$$

It is trivial to see that

$$u_{\mu}\tilde{Q}^{\mu\nu} = 0 = u_{\nu}\tilde{Q}^{\mu\nu}$$

### 7.2.2 Shear, Stress and Expansion

The 3-velocity of the line of congruence $\xi^i + \delta\xi^i$ relative to $\xi^i$ is defined as

$$v^{\mu} = h^{\mu}{}_{\nu}\dot{\eta}^{\nu}$$

(3-velocity since $v^{\mu}u_{\mu} = 0$ implies only 3 independent components)

**Lemma 7.2(a)**

$v^{\mu} = A^{\mu}{}_{\nu}\eta^{\nu}$ where $A_{\mu\nu} = u_{\mu;\lambda}h^{\lambda}{}_{\nu}$

**Proof**

$$
\begin{aligned}
v^\mu &= h^\mu{}_\nu \dot{\eta}^\nu \\
\eta^\nu &= h^\nu{}_\lambda \zeta^\lambda \\
&= (\delta^\nu_\lambda + u^\nu u_\lambda)\zeta^\lambda \\
&= \zeta^\nu + u^\nu u_\lambda \zeta^\lambda \\
\Rightarrow \dot{\eta}^\nu &= \dot{\zeta}^\nu + \dot{u}^\nu u_\lambda \eta^\lambda + u^\nu (u_\lambda \zeta^\lambda)^{\cdot} \\
&= u^\nu{}_{;\lambda}\zeta^\lambda + \dot{u}^\nu u_\lambda \zeta^\lambda + u^\nu (u_\lambda \zeta^\lambda)^{\cdot} \\
\Rightarrow v^\mu &= h^\mu{}_\nu \dot{\eta}^\nu \\
&= h^\mu{}_\nu u^\nu{}_{;\lambda}\zeta^\lambda + h^\mu{}_\nu \dot{u}^\nu u_\lambda \zeta^\lambda + h^\mu{}_\nu u^\nu (u_\lambda \zeta^\lambda)^{\cdot} \\
&= (\delta^\mu_\nu + u^\mu u_\nu) u^\mu{}_{;\lambda}\zeta^\lambda + (\delta^\mu_\nu + u^\mu u_\nu)\dot{u}^\nu u_\lambda \zeta^\lambda \\
&= u^\mu{}_{;\lambda}\zeta^\lambda + u^\mu (u_\nu u^\nu{}_{;\lambda})\zeta^\lambda + u^\mu{}_{;\rho} u^\rho u_\lambda \zeta^\lambda + u^\mu u_\nu u^\nu{}_{;\rho} u^\rho u_\lambda \zeta^\lambda \\
&= u^\mu{}_{;\lambda}\zeta^\lambda + u^\mu{}_{;\rho} u^\rho u_\lambda \zeta^\lambda \\
&= u^\mu{}_{;\rho}\delta^\rho_\lambda \zeta^\lambda + u^\mu{}_{;\rho} u^\rho u_\lambda \zeta^\lambda \\
&= u^\mu{}_{;\rho}(\delta^\rho_\lambda + u^\rho u_\lambda)\zeta^\lambda \\
&= u^\mu{}_{;\rho} h^\rho{}_\lambda \zeta^\lambda \\
&= u^\mu{}_{;\rho}\eta^\rho \\
&= u^\mu{}_{;\rho} h^\rho{}_\nu \eta^{nu} \\
&= A^\mu{}_\nu \eta^\nu
\end{aligned}
$$

where $A^\mu{}_\nu = u^\mu{}_{;\rho} h^\rho{}_\nu$
$\iff \; A_{\mu\nu} = u_{\mu;\rho} h^\rho{}_\nu \quad \square$

**Lemma 7.2(b)**

$A_\mu$ may be written as

$$A_{\mu\nu} = u_{\mu;\lambda} h^\lambda{}_\nu = \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu}$$

where

$$\sigma_{\mu\nu} = u_{(\mu;\nu)} + \dot{u}_{(\mu} u_{\nu)} - \frac{1}{3} u^\lambda{}_{;\lambda} h_{\mu\nu}$$

is a symmetric, trace free $\binom{0}{2}$ tensor known as the shear tensor.

$$\omega_{\mu\nu} = u_{[\mu;\nu]} + \dot{u}_{[\mu} u_{\nu]}$$

is an antisymmetric $\binom{0}{2}$ tensor known as the vorticity tensor. $\theta = u^\lambda{}_{;\lambda}$ is the expansion or contraction of the congruence.

**Proof**

$$\begin{aligned}\text{RHS} &= u_{(\mu;\nu)} + \dot{u}_{(\mu}u_{\nu)} - \frac{1}{3}u^{\lambda}{}_{;\lambda}h_{\mu\nu} + u_{[\mu;\nu]} + \dot{u}_{[\mu}u_{\nu]} + \frac{1}{3}u^{\lambda}{}_{;\lambda}h_{\mu\nu} \\ &= u_{\mu;\nu} + \dot{u}_{\mu}u_{\nu} \\ &= u_{\mu;\lambda}\delta^{\lambda}_{\nu} + u^{\lambda}u_{\mu;\lambda}u_{\nu} \\ &= u_{\mu;\lambda}(\delta^{\lambda}_{\nu} + u^{\lambda}u_{\nu}) \\ &= u_{\mu;\lambda}h^{\lambda}{}_{\nu} = A_{\mu\nu} \quad \square\end{aligned}$$

**Conclusion**

In going from $\tau$ to $\tau + \delta\tau$ along $\xi^i$, the 3-space of orthogonal connecting vectors undergoes a linear transformation or

1. a shear, or distortion
2. a twist, or rotation
3. an expansion, or contraction.

## 7.3 The Friedman-Robertson-Walker (FRW) Metric

### 7.3.1 Isotropy

The model universe is a spacetime manifold $(M, g)$ filled with a congruence $K$ of timelike worldlines $k \in K$. We assume spatial isotropy.

**Definition:** If $(M, g)$ is isotropic with respect to a $k \in K$, then all directions orthogonal to $k$ at each point of $k$ are equivalent, i.e. if $\Omega$ is a hypersurface orthogonal to the tangent to $k$, then there are no preferred directions in $\Omega$.

**Consequence of isotropy of $k$**

1. $\dot{u}^{\mu} = 0$ along $k$, i.e. $k$ is a geodesic.

   **Proof**

$$\begin{aligned} u^{\mu}u_{\mu} &= -1 \\ \Rightarrow u^{\mu}\dot{u}_{\mu} &= 0\end{aligned}$$

$\Rightarrow$ at each point of $k, \dot{u}^\mu$ is a special vector orthogonal to $u^\mu$.
Isotropy $\Rightarrow$ no such vector exists $\Rightarrow u^\mu = 0$.

2. $\sigma_{\mu\nu} = 0$ along $k$.

**Proof**

$\sigma_{\mu\nu}$ is a $4x4$ symmetric trace-free matrix which is orthogonal to $u^\mu$ in both of its indices.

$$\sigma_{\mu\nu} u^\mu = \sigma_{\mu\nu} u^\nu = 0$$

$\sigma_{\mu\nu}$ has 4 mutually orthogonal eigenvectors, one timelike and 3 spacelike, where $u^\mu$ is the unit timelike eigenvector with eigenvalue 0. Hence, the 3 unit spacelike eigenvectors constitute 3 unique directions orthogonal to $u^\mu$ (i.e. lying in the hypersurface $\Omega$) at each point of $k$. By isotropy, no such directions $\Rightarrow \sigma_{\mu\nu} = 0$.

3. $\omega_{\mu\nu} = 0$ along $k$.

**Proof**

Let $\tilde{\varepsilon}_{\mu\nu\lambda\rho}$ be the Levi-Civita symbol and take $\varepsilon_{\mu\nu\lambda\rho} = \sqrt{-g}\tilde{\varepsilon}_{\mu\nu\lambda\rho}$ as the Levi-Civita tensor. We define the vorticity vector by

$$\omega_\mu = \varepsilon_{\mu\nu\lambda\rho} u^\nu \omega^{\lambda\rho}$$

$\omega_\mu u^\mu = 0$. There is a unique vector orthogonal to $u^\mu$.
Isotropy $\Rightarrow \omega_\mu = 0 \Rightarrow \omega_{\mu\nu} = 0$.

4. $h^\mu \theta_{,\mu} = 0$ along $k$.

**Proof**

$u_\mu (h^\mu{}_\nu \theta_{,\mu}) = 0$, hence $h^\mu{}_\nu \theta_{,\mu}$ is a unique vector orthogonal to $u^\mu$.
Isotropy $\Rightarrow h^\mu{}_\nu \theta_{,\mu} = 0$.

**Definition:** Isotropic universe:$(M, g)$ is spatially isotropic if it is isotropic with respect to all $k \in K$. Since a worldline $k$ passes through every point of $M$, this implies:

$$\begin{aligned} \dot{u}^\mu &= 0 \\ \sigma^{\mu\nu} &= 0 \\ \omega^{\mu\nu} &= 0 \\ h^\mu{}_\nu \theta_{,\mu} &= 0 \end{aligned}$$

### 7.3.2 Comoving Coordinates

Isotropy implies

$$\omega_{\mu\nu} = u_{[\mu;\nu]} + \dot{u}_{[\mu}u_{\nu]} = 0$$

and

$$\dot{u}^\mu = 0$$

$$\Rightarrow u_{[\mu;\nu]} = 0$$

$$\iff u_{\mu,\nu} = u_{\nu,\mu}$$

$$\iff \text{there exists a scalar function } t(x^\mu) \text{ such that } u_\mu = -t_{,\mu}$$

$$\iff u_\mu dx^\mu = -t_{,\mu}dx^\mu = -dt$$

$\Rightarrow$ the worldlines $k \in K$ intersect the $t$ =constant hypersurfaces orthogonally.

Since

$$u^\mu = \frac{\partial x^\mu}{\partial \tau}$$

and

$$\begin{aligned}\frac{dt}{d\tau} &= \frac{\partial t}{\partial x^\mu}\frac{\partial x^\mu}{\partial \tau}\\ &= t_{,\mu}u^\mu\\ &= -u_\mu u^\mu\\ &= 1\end{aligned}$$

$$\Rightarrow \tau = t \qquad \text{(taking the integration constant to be zero)}$$

Hence $t$ is the proper time along each $k \in K$, this is known as cosmic time.

Taking $\{x^\mu\} = (t, x^i)$ to be the coordinates on $M$, then $x^i$ =constant label each $k \in K$. In these coordinates

$$u^\mu = \frac{\partial x^\mu}{\partial \tau} = \frac{\partial x^\mu}{\partial t} = \delta^\mu_t$$

$$\Rightarrow u^\mu = (1, 0, 0, 0).$$

Since the 4-velocity in these coordinates has no spatial components, we say the coordinates are "co-moving" with the matter.

Galaxies move on integral curves of the vector field $\frac{\partial}{\partial t}$.

### 7.3.3 The Spacetime Metric

We have shown that isotropy yields

$$u_\mu g_{\mu\nu} u^\nu = -t_{,\mu}$$

which, in co-moving coordinates, implies

$$\begin{aligned} g_{\mu\nu}\delta^\nu_t &= -t_{,\mu} = -\delta^t_\mu \\ \Rightarrow g_{\mu t} &= -\delta^t_\mu \\ \Rightarrow g_{tt} &= -\delta^t_t = -1 \\ g_{ti} &= 0 \quad i = 1, 2, 3. \end{aligned}$$

The metric reads

$$ds^2 = -dt^2 + g_{ij}(t, x^i)dx^i dx^j$$

Next we consider

$$h^\mu{}_\nu \theta_{,\mu} = 0$$

where

$$h^\mu{}_\nu = \delta^\mu_\nu + u^\mu u_\nu$$

wich in comoving coordinates gives

$$\begin{aligned} &(\delta^\mu_\nu - \delta^\mu_t \delta^t_\nu)\theta_{,\mu} = 0 \\ &\Rightarrow \theta_{,\nu} = \delta^t_\nu \theta_{,t} \\ &\Rightarrow \theta_{,i} = 0 \\ &\Rightarrow \theta = \theta(t) \end{aligned}$$

Finally, we have that

$$\sigma_{\mu\nu} = 0$$

The orthogonal connecting vector of two neighbouring galaxies $x^\mu$ and $x^\mu + \delta x^\mu$ in comoving coordinates is

$$\eta^\mu = (0, \delta x^i).$$

Let $l$ be the invariant length of this vector

$$l^2 = g_{\mu\nu}\eta^\mu \eta^\nu = g_{ij}\delta x^i \delta x^j \qquad (i, j = 1, 2, 3)$$

$$\Rightarrow \boxed{2l\dot{l} = \frac{\partial g_{ij}}{\partial t}\delta x^i \partial x^j} \qquad \text{(A)}$$

We may also write $\eta^\mu = l n^\mu$ where $n^\mu$ is a unit spacelike vector. Recall that

$$h^\mu{}_\nu \dot{\eta}^\nu = A^\mu{}_\nu \eta^\nu$$

where

$$A^\mu{}_\nu = \sigma^\mu{}_\nu + \omega^\mu{}_\nu + \frac{1}{3}\theta h^\mu{}_\nu$$

In our case $\sigma_{\mu\nu} = 0$.

$$\Rightarrow h^\mu_\nu \dot{\eta}^\nu = \omega^\mu{}_\nu \eta^\nu + \frac{1}{3}\theta h^\mu_\nu \eta^\nu$$

$$\Rightarrow h^\mu_\nu \dot{l} n^\nu + h^\mu_\nu l \dot{n}^\nu = \omega^\mu{}_\nu l n^\nu + \frac{1}{3}\theta h^\mu_\nu l n^\nu$$

Multiplying by $n_\mu$ and using $n^\mu n_\mu = 1$ and $n_\mu u_\mu = 0$

$$\dot{l} + (\delta^\mu_\nu + u^\mu u_\nu) l n_\mu \dot{n}^\nu = \omega^\mu{}_\nu l n_\mu n_\nu + \frac{1}{3}\theta(\delta^\mu_\nu + u^\mu u_\nu) l n_\mu n^\nu$$

$$\delta^\mu_\nu + u^\mu u_\mu = 0 \qquad \text{(since } \dot{n}^\nu n_\nu = 0\text{)}$$

$$\omega^\mu{}_\nu l n_\mu n^\nu = 0 \qquad \text{(since } \omega^\mu{}_\nu \text{ anti-symmetric, } n_\mu n^\nu \text{ symmetric)}$$

$$\Rightarrow \boxed{\dot{l} = \frac{1}{3}\theta l} \tag{B}$$

where $\theta = \theta(t)$ in comoving coordinates.

Subbing $(B)$ into $(A)$

$$\frac{2}{3}\theta(t) l^2 = \frac{\partial g_{ij}}{\partial t}\delta x^i \delta x^j$$

$$\Rightarrow \frac{2}{3}\theta(t) g_{ij}\delta x^i \delta x^j = \frac{\partial g_{ij}}{\partial t}\delta x^i \delta x^j$$

$$\Rightarrow \frac{2}{3}\theta(t) g_{ij} = \frac{\partial g_{ij}}{\partial t}$$

A separable solution of the form $g_{ij} = h_{ij}(x^i) l^2(t)$ satisfies this equation, so the metric now reads

$$ds^2 = dt^2 + l^2(t) h_{ij}(x^i) dx^i dx^j$$

where $l(t)$ satisfies

$$\dot{l} = \frac{1}{3}\theta(t) l$$

and $h_{ij}$ is a positive definite metric on a Riemannian 3-manifold which is isotropic at each of its points. This is consistent with asserting that $h_{ij}$ is a positive definite 3-metric of *constant curvature*. There are only 3 distinct possibilities: $\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3$.

1. Flat Space $\mathbb{R}^3$: This is the familiar infinite Euclidean geometry

$$\begin{aligned} {}^{(3)}dx^2 &= h_{ij}dx^i dx^j \\ &= dx^2 + dy^2 + dz^2 \\ &= dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \end{aligned}$$

2. Three-Sphere $\mathbb{S}^3$: The compact space of constant curvature. To visualise this, we embed $\mathbb{S}^3$ in $\mathbb{R}^4$ with points on $\mathbb{S}^3$ satisfying

$$x^2 + y^2 + z^2 + w^2 = a^2$$

$$\begin{aligned} \Rightarrow^{(3)} ds^2 &= dx^2 + dy^2 + dz^2 + dw^2 \\ &= dx^2 + dy^2 + dz^2 + \frac{(xdx + ydy + zdz)^2}{a^2 - x^2 - y^2 - z^2} \end{aligned}$$

which, in spherical polar coordinates, yields

$$\begin{aligned} {}^{(3)}ds^2 &= dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{r^2dr^2}{a^2 - r^2} \\ &= \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \end{aligned}$$

$$0 \leq r \leq a$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \varphi \leq 2\pi$$

3. Hyperbolic Space $\mathbb{H}^3$: The infinite space of constant negative curvature. To visualise this, we embed $\mathbb{H}^3$ in a 4 dimensional Lorentzian space

$$x^2 + y^2 + z^2 - w^2 = -a^2$$

which yields

$$^{(3)}ds^2 = \frac{dr^2}{1 + \frac{r^2}{a^2}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

Scaling the 'radial' variable

$$r \to ra$$

then we may rewrite the general form of the 3pmetric of constant curvature as

$$^{(3)}ds^2 = a^2\Big[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\Big]$$

where

$$\begin{aligned} k = +1 \quad & \mathbb{S}^3 \quad \text{(closed)} \\ 0 \quad & \mathbb{R}^3 \quad \text{(flat,open)} \\ -1 \quad & \mathbb{H}^3 \quad \text{(open)} \end{aligned}$$

The 4D metric in comoving coordinates is therefore

$$ds^2 = -dt^2 + l^2(t)a^2\Big[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\Big]$$

Finally, taking $a(t) = l(t)a$

$$\boxed{ds^2 = -dt^2 + a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\Big]}$$

The Friedmann-Robertson-Walker metric, where $a(t)$ satisfies

$$\frac{\dot{a}}{a} = \frac{1}{3}\Theta(t)$$

and is known as the scale factor.

Another particularly useful form of the FRW metric is obtained by the transformation

$$d\chi^2 = \frac{dr^2}{1-kr^2}$$

$$\Rightarrow \boxed{ds^2 = -dt^2 + a^2(t)[d\chi^2 + f^2(\chi)(d\theta^2 \sin^2\theta d\varphi^2)]}$$

where

$$\begin{aligned} f(\chi) = \sin\chi \quad & 0 < \chi < \pi \quad \mathbb{S}^3 \\ \chi \quad & 0 \leq \chi\infty \quad \mathbb{R}^3 \\ \sinh\chi \quad & 0 \leq \chi < \infty \quad \mathbb{H}^3 \end{aligned}$$

The scale factor $a(t)$ is determined by the Einstein field equations.

## 7.4 Cosmological Red-Shift and Hubble's Law

Consider light emitted by a galaxy with world line

$$r + r_0, \theta = \theta_0, \varphi = \varphi_0$$

Suppose this light is received by our galaxy whose world line is

$$r + r_1, \theta = \theta_1, \varphi = \varphi_1$$

Along photon worldline $P_0P_1$ and $Q_0Q_1$

$$ds = 0, \theta = \theta_0, \varphi = \varphi_0$$

$$\Rightarrow \frac{dt^2}{a^2(t)} = \frac{dr^2}{1 - kr^2}$$

Assuming, without loss of generality, that $r_1 > r_0$ and $a(t) > 0$

$$\Rightarrow \frac{dt}{a(t)} = \frac{dr}{\sqrt{1 - kr^2}}$$

If $n$ photons are emitted by $r = r_0$ in proper time $dt_0$, and received by $r = r_1$ in proper time $dt$, then the emitted frequency is $\nu_0 = \frac{n}{dt_0}$ while the received frequency is $\nu_1 = \frac{n}{dt_1}$

$$\therefore \frac{\nu_0}{\nu_1} = \frac{\lambda_1}{\lambda_0} = \frac{dt_1}{dt_0}$$

Integrating along $P_0P_1$ and $Q_0Q_1$ gives

$$\int_{t_0}^{t_1} \frac{dt}{a(t)} = \int_{r_0}^{r_1} \frac{dr}{\sqrt{1 - kr^2}}$$

and

$$\int_{t_0+dt_0}^{t_1+dt_1} \frac{dt}{a(t)} = \int_{r_0}^{r_1} \frac{dr}{\sqrt{1 - kr^2}}$$

$$\Rightarrow \int_{t_0}^{t_1} \frac{dt}{a(t)} = \int_{t_0+dt_0}^{t_1+dt_1} \frac{dt}{a(t)}$$

writing

$$a^{-1}(t) = \frac{df(t)}{dt}$$

$$\Rightarrow f(t_1) - f(t_0) - f(t_1 + dt_1) + f(t_0 + dt_0) = 0$$

Taylor expanding, keeping only first order terms

$$\left(\frac{df}{dt}\right)_{t=t_1} dt_1 + \left(\frac{df}{dt}\right)_{t=t_0} dt_0 = 0$$

$$\Rightarrow \boxed{\frac{dt_1}{dt_0} = \frac{a(t_1)}{a(t_0)} = \frac{\lambda_1}{\lambda_0}}$$

Observations of distant galaxies$\Rightarrow$ a red-shift

$$\begin{aligned} &\lambda_1 > \lambda_0 \\ \Longleftrightarrow\ &a(t_1) > a(t_0) \\ \Longleftrightarrow\ &\dot{a} > 0 \qquad \text{(in the present epoch)} \\ \therefore\ &\Theta = \frac{3\dot{a}}{a} > 0 \end{aligned}$$

$\Rightarrow$ the universe is expanding.

Galaxies are receeding away from one another at a rate proportional to the distance between them-Hubble's Law (1929).

Hubble Parameter

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

## 7.5 Einstein's Equations

The scale factor $a(t)$ is determined by considering Einstein's field equations with a cosmological constant term

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

### 7.5.1 Matter Content of the Universe

Consistent with spatial homogeneity and isotropy in a comoving frame with $n^\mu = (1,0,0,0)$ the e.m.s. tensor takes the form of a perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$$

Also, isotropy $\Rightarrow \rho = \rho(t), p = p(t)$.

Conservation of e.m. $T^{\mu\nu}{}_{,\nu} = 0$ gives constraint

$$\dot{\rho} + \frac{3\dot{a}}{a}(\rho + p) = 0$$

where

$$\begin{aligned} \rho &= \text{proper energy density} \\ p &= \text{isotropic pressure} \end{aligned}$$

To solve this, we further require an equation of state, we assume a barotropic fluid of the form

$$P = (\gamma - 1)\rho \qquad (\gamma = \text{ a constant})$$

Two common cases are matter/radiation domination:

1. Dust: $\gamma = 1, p = 0$, i.e. pressureless not interacting matter

$$\Rightarrow \frac{\dot{\rho}}{\rho} = \frac{-3\dot{a}}{a}$$
$$\Rightarrow \rho \propto a^{-3}$$
$$\Rightarrow \rho = \frac{\rho_0 a_0^3}{a^3}$$

2. Radiation: $\gamma = \frac{4}{3}, p = \frac{\rho}{3}$

$$\Rightarrow \frac{\dot{\rho}}{\rho} = \frac{-4\dot{a}}{a}$$
$$\Rightarrow \rho \propto a^{-4}$$
$$\Rightarrow \rho = \frac{\rho_0 a_0^4}{a^4} \qquad \text{(dilution of energy to expansion and redshift)}$$

### 7.5.2 The Friedman and Raychauduri Equations

Non-zero components of the FRW-metric:

$$R_{tt} = -3\frac{\ddot{a}}{a}$$

$$R_{ij} = \Big[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2}\Big]g_{ij}$$
$$R = g^{tt}R_{tt} + g^{ij}R_{ij}$$
$$= 6\Big[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\Big]$$

- The $tt$-component of Einstein's equations

$$R_H - \frac{1}{2}Rg_{tt} + \Lambda g_{tt} = 8\pi T_{tt}$$
$$\Rightarrow -3\frac{\ddot{a}}{a} + 3\Big[\left(\frac{\ddot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\Big] - \Lambda = 8\pi\rho$$
$$\Rightarrow \boxed{\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi\rho}{3}} \qquad \text{(Friedman Equation)}$$

- $ij$-component

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \Lambda = -8\pi p$$

Subtracting Friedman's equations

$$\boxed{\frac{\ddot{a}}{a} - \frac{\Lambda}{3} = \frac{-4\pi}{3}(\rho + 3p)} \qquad \text{(Raychauderi Equation)}$$

Standard cosmological models are obtained by solving these for some equation of state.

## 7.6 Cosmological Models with Vanishing $\Lambda$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi\rho}{3}$$

$$\left(\frac{\ddot{a}}{a}\right) = \frac{-4\pi}{3}(\rho + 3p)$$

### 7.6.1 Geometry of the Universe

Critical density $\rho_c$: energy density of flat universe

$$\rho_c = \frac{3}{8\pi}\left(\frac{\dot{a}}{a}\right)^2 = \frac{dH^2}{8\pi} \qquad \text{(in theory, } H_0 \text{ is measurable)}$$

If we now define the density parameter of the universe to be

$$\Omega = \frac{\rho}{\rho_c}$$

Friedman

$$\Rightarrow \frac{k}{a^2 H} = \Omega - 1$$

$$\begin{aligned} k = +1 &\quad \Omega > 1 \quad \text{(closed)} \\ 0 &\quad \Omega = 1 \quad \text{(flat,open)} \\ -1 &\quad \Omega < 1 \quad \text{(open)} \end{aligned}$$

The actual geometry is still hotly debated.

### 7.6.2 Big Bang Singularities

Assuming the matter content of the universe obeys the strong energy condition (SEC)

$$T_{\mu\nu}u^\mu u^\nu \geq -\frac{1}{2}T$$

for any time-like vector $u^\mu$ and $T = T^\mu{}_\mu$.
In our FRW metric with perfect fluid matter distribution, this implies

$$\rho + 3P \geq 0$$

This in turn implies (from the Raychaudhuri equation)

$$\begin{aligned}
\frac{\ddot{a}}{a} &\leq 0 \\
\iff \dot{H} + H^2 &\leq 0 \\
\Rightarrow \int \frac{dH}{H^2} &\leq -\int dt \\
\Rightarrow \frac{-1}{H} + \frac{1}{H_0} &\leq -(t - t_0) \\
\Rightarrow H &\leq \frac{1}{H_0^{-1} + (t - t_0)}
\end{aligned}$$

Using the fact that $H = \frac{\dot{a}}{a}$ and integrating again, we get

$$\int \frac{da}{a} \leq \int \frac{1}{H_0^{-1} + (t - t_0)} dt$$

$$\Rightarrow \boxed{a(t) \leq a(t_0) \left( \frac{H_0^{-1} + (t - t_0)}{H_0^{-1}} \right)}$$

i.e. $a(t)$ is bounded above by a linear function of $t$, which has $t$-intercept at $t = t_0 - h_0^{-1}$.
For some finite time $t > t_0 - H_0^{-1}$, we must have $a(t) = 0$. But as $a(t) \to 0, \rho \to \infty$, we have an infinite energy state, and all known laws of physics breaks down. (Singularity Theorems-Hawking,Penrose,Geroch)
All FRW moels with $\Gamma = 0$ with a matter distribution satisfying the SEC predict a 'Big Bang' singularity at some finite time in the past.

### 7.6.3 The Fate of the Universe (Eschatology)

(Matter Domination $P = 0$).
We define a conformal time $\tau$ by

$$d\tau = \frac{dt}{a}$$

and we denote derivatives with respect to conformal time by ', i.e.

$$a' = \frac{da}{d\tau}$$

Defining $\mathcal{H} = \frac{a'}{a}$, then

$$H = \frac{\dot{a}}{a} = \frac{a'}{a}\frac{d\tau}{dt} = \frac{\mathcal{H}}{a}$$

The Friedmann equation in conformal time becomes

$$\mathcal{H}^2 + k = \frac{8\pi\rho a^2}{3}$$

and the Raychaudhuri equation becomes

$$\frac{\mathcal{H}'}{a^2} = \frac{-4\pi\rho}{3} \qquad \text{(for dust } P = 0\text{)}$$

Combining the two to eliminate $\rho$, we obtain

$$\boxed{2\mathcal{H}' + \mathcal{H}^2 + k = 0}$$

For an open universe, $k = -1$: We have

$$\int \frac{d\mathcal{H}}{1 - \mathcal{H}^2} = \frac{1}{2}\int d\tau$$

This may be solved with the substitution

$$\begin{aligned} \mathcal{H} &= \coth\left(\frac{u}{2}\right) \\ \Rightarrow \mathcal{H} &= \coth\left(\frac{\tau}{2}\right) \\ \Rightarrow \mathcal{H}' &= \frac{-1}{2\sinh^2}\left(\frac{\tau}{2}\right) = \frac{1}{1 - \cosh\tau} \end{aligned}$$

From the Raychaudhuri equation

$$\frac{\mathcal{H}'}{a} = \frac{-4\pi\rho}{3} = \frac{-4\pi\rho_0 a_0^3}{3a^3}$$

$$\Rightarrow a(\tau) = \frac{-4\pi}{3}\rho_0 a_0^3(1-\cosh\tau)$$

We have

$$\Omega_0 = \frac{8\pi\rho_0}{3H_0^2}$$

$$\Rightarrow a(\tau) = -\frac{1}{2}\Omega_0 H_0^2 a_0^3(1-\cosh\tau)$$

We also have

$$a_0^2 H_0^2 = \frac{k}{\Omega_0 - 1}$$

$$\Rightarrow a_0^3 = H_0^{-3}(1-\Omega)^{-\frac{3}{2}})$$

$$\boxed{a(\tau) = \frac{1}{2}\Omega_0 H_0^{-1}(1-\Omega_0)^{-\frac{3}{2}}(\cosh\tau - 1)} \qquad \text{(A)}$$

We also have

$$dt = a d\tau$$

$$\Rightarrow t = \frac{1}{2}\Omega_0 H_0^{-1}(1-\Omega_0)^{-\frac{3}{2}}\int(\cosh\tau - 1)d\tau$$

$$\boxed{t = \frac{1}{2}\Omega_0 H_0^{-1}(1-\Omega_0)^{-\frac{3}{2}}(\sinh\tau - \tau)} \qquad \text{(B)}$$

$(A)$ and $(B)$ together form a parametric solution for an open $k=-1$ universe. As $t \to \infty$, the $k=-1$ universe expands forever. As $a(t)$ gets larger and larger, in the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2} = \frac{8\pi\rho_0 a_0^3}{3a^3}$$

the curvature term $\frac{1}{a^3}$ dominates the matter term

$$\rho \propto a^{-3} \ll \frac{|k|}{a^2} \qquad \text{(for } a \to \infty\text{)}$$

$$\Rightarrow \frac{\dot{a}}{a} \sim \frac{1}{a^2}$$

$$\Rightarrow a(t) \propto t$$

The universe undergoes a period of free expansion as the density drops off rapidly.

$$\rho \propto t^{-3}$$

# 8 The Linear Approximation

## 8.1 The Einstein Equations in the Linear Approximation

We begin with the assumption that there exists coordinates in which the metric of a weak gravitational field can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where

$$|h_{\mu\nu}| \ll 1$$

Introduce a fictitious 'book-keeping' parameter

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$$

We neglect all $O(\epsilon^2)$ and higher terms. We further assume the boundary condition

$$\lim_{r\to\infty} h_{\mu\nu} = 0$$

i.e. the spacetime is asymptotically flat.

If we think of $h_{\mu\nu}$ as a tensor on $\eta_{\mu\nu}$, then we can raise indices of $h_{\mu\nu}$ by contracting with $\eta^{\mu\nu}$, e.g.

$$h_{\mu\nu} = \eta^{\mu\nu}\eta^{\nu\sigma}h_{\lambda\sigma}$$

It is easy to verify that the metric inverse (to first order) is

$$g^{\mu\nu} = \eta^{\mu\nu} - \epsilon h^{\mu\nu}$$

To see this, we check that

$$g^{\mu\nu}g_{\nu\lambda} = \delta^\mu_\lambda + O(\epsilon^2)$$

. The Christoffell symbols in the linear approximation are

$$\begin{aligned}
\Gamma^\mu{}_{\nu\lambda} &= \frac{1}{2}g^{\mu\sigma}(g_{\sigma\nu,\lambda} + g_{\lambda\sigma,\nu} - g_{\nu\lambda,\sigma}) \\
&= \frac{1}{2}(\eta^{\mu\sigma} - \epsilon h^{\mu\sigma})(\epsilon h_{\sigma\nu,\lambda} + \epsilon h_{\lambda\sigma,\nu} - \epsilon h_{\nu\lambda,\sigma}) \\
&= \frac{1}{2}\epsilon(h^\mu{}_{\nu,\lambda} + h^\mu{}_{\lambda,\nu} - h_{\nu\lambda,}{}^\mu) + O(\epsilon^2)
\end{aligned}$$

The Riemann curvature tensor is given by

$$\begin{aligned}
R_{\mu\nu\lambda\sigma} &= g_{\mu\rho}R^{\rho}{}_{\nu\lambda\sigma} \\
&= (\eta_{\mu\rho} + \epsilon h_{\mu\rho})(\Gamma^{\rho}{}_{\nu\sigma,\lambda} - \Gamma^{\rho}{}_{\nu\lambda,\sigma} + \Gamma^{\alpha}{}_{\nu\sigma}\Gamma^{\rho}{}_{\alpha\lambda} - \Gamma^{\alpha}{}_{\nu\lambda}\Gamma^{\rho}{}_{\alpha\sigma}) \\
&= \eta_{\nu\rho}\Big[\frac{1}{2}\epsilon(h^{\rho}{}_{\nu,\sigma\lambda} + h^{\rho}{}_{\sigma,\nu\lambda} - h_{\nu\sigma,}{}^{\rho}{}_{\lambda}) - \frac{1}{2}\epsilon(h^{\rho}{}_{\nu,\lambda\sigma} + h^{\rho}{}_{\lambda,\nu\sigma} - h_{\nu\lambda,}{}^{\rho}{}_{\sigma})\Big] \\
&= \frac{1}{2}\epsilon(h_{\mu\sigma,\nu\lambda} + h_{\nu\lambda,\nu\sigma} - h_{\mu\lambda,\nu\sigma} - h_{\nu\sigma,\mu\lambda}) + O(\epsilon^2)
\end{aligned}$$

The Ricci tensor components in the linear approximation are

$$\begin{aligned}
R_{\nu\sigma} &= g^{\mu\lambda}R_{\mu\nu\lambda\sigma} \\
&= \eta^{\mu\lambda}\frac{1}{2}\epsilon(h_{\mu\sigma,\nu\lambda} + h_{\nu\lambda,\nu\sigma} - h_{\mu\lambda,\nu\sigma} - h_{\nu\sigma,\mu\lambda}) + O(\epsilon^2) \\
&= \frac{1}{2}\epsilon(h^{\lambda}{}_{\sigma,\nu\lambda} + h^{\mu}{}_{\nu,\mu\sigma} - h_{,\nu\sigma} - \Box h_{,\nu\sigma}) + O(\epsilon^2)
\end{aligned}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$ and

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_\mu\partial^\mu = \frac{-\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The Ricci scalar in the linear approximation is

$$\begin{aligned}
R &= (\eta^{\mu\nu} - \epsilon h^{\mu\nu})\frac{1}{2}\epsilon(h^{\lambda}{}_{\nu,\mu\lambda} + h^{\rho}{}_{\mu,\rho\nu} - h_{,\mu\nu} - \Box h_{\mu\nu}) + O(\epsilon^2) \\
&= \epsilon(h^{\mu\nu}{}_{,\mu\nu} - \Box h)
\end{aligned}$$

Finally, the Einstein tensor in the linear approximation is

$$\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \\
&= \frac{1}{2}\epsilon(h^{\lambda}{}_{\nu,\mu\lambda} + h^{\lambda}{}_{\mu,\lambda\nu} - h_{,\mu\nu} - \Box h_{\mu\nu}) - \frac{1}{2}\eta_{\mu\nu}\epsilon(h^{\lambda\rho}{}_{,\lambda\rho} - \Box h) + O(\epsilon^2) \\
\Rightarrow G_{\mu\nu} &= \frac{1}{2}\epsilon(h^{\lambda}{}_{\nu,\mu\lambda} + h^{\lambda}{}_{\mu,\lambda\nu} - h_{,\mu\nu} - \Box h_{\mu\nu} - \eta_{\mu\nu}h^{\lambda\rho}{}_{,\lambda\rho} + \eta_{\mu\nu}\Box h)
\end{aligned}$$

The linearised vacuum field equations would involve setting this to be zero and solving for $h_{\mu\nu}$. It is convenient to write the Einstein tensor in terms of a new dependent tensor. the "star conjugate" of $h_{\mu\nu}$

$$h^*_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{1}$$

$$h^*_{\mu\nu} = h^*_{\nu\mu}$$

Note that

$$h^* = \eta^{\mu\nu} h^*_{\mu\nu} = h - \frac{1}{2}(4)h = -h$$

Using this to invert (1)

$$h_{\mu\nu} = h^*_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^*$$

Note also, we can perform a double star conjugate

$$h^{**}_{\mu\nu} = h_{\mu\nu}$$

In terms of $h^*_{\mu\nu}$, the Einstein tensor reads

$$\begin{aligned} G_{\mu\nu} =& \frac{1}{2}\epsilon\Big[(h^{*\lambda}{}_{\nu} - \frac{1}{2}\delta^\lambda_\nu h^*)_{,\mu\lambda} + (h^{*\lambda}{}_{\mu} - \frac{1}{2}\delta^\lambda_\mu h^*)_{,\lambda\nu} + h^*_{,\mu\nu} \\ & - \Box(h^*_{\mu\nu}\frac{1}{2}\eta_{\mu\nu}h^*) - \eta_{\mu\nu}(h^{*\lambda\rho} - \frac{1}{2}\eta^{\lambda\rho}h^*)_{,\lambda\rho} - \eta_{\mu\nu}\Box h^*)\Big] \end{aligned}$$

$$\Rightarrow \boxed{G_{\mu\nu} = \frac{1}{2}\epsilon\Big[-\Box h^*_{\mu\nu} + h^{*\lambda}{}_{\mu,\lambda\nu} + h^{*\lambda}{}_{\nu,\lambda\mu} - \eta_{\mu\nu}h^{*\lambda\rho}{}_{,\lambda\rho}\Big]}$$

## 8.2 Gauge Transformations

Let us consider coordinate transformations of the form

$$x^\mu \to \hat{x}^\mu = x^\mu + \epsilon\xi^\mu$$

(Sacrificed general covariance, considering only coordinate transformations close to the identity).
Under this transformation

$$\hat{g}_{\mu\nu} = \frac{\partial x^\lambda}{\partial \hat{x}^\mu}\frac{\partial x^\sigma}{\partial \hat{x}^\nu}\dot{g}_{\lambda\sigma}$$

But

$$\begin{aligned} x^\mu &= \hat{x}^\mu - \epsilon\xi(x^\nu) \\ &= \hat{x}^\lambda - \epsilon\xi(\hat{x}^\mu - \epsilon\xi(x^\mu)) \\ &= \hat{x}^\mu + \epsilon\xi^\mu(\hat{x}) + O(\epsilon^2) \\ \therefore \frac{\partial x^\mu}{\partial \hat{x}^\nu} &= \delta^\mu_\nu - \epsilon\xi^\mu{}_{,\nu} + O(\epsilon^2) \\ \Rightarrow \hat{g}_{\mu\nu} &= g_{\mu\nu} - \epsilon\xi_{\mu,\nu} - \epsilon\xi_{\nu,\mu} + O(\epsilon^2) \end{aligned}$$

and since

$$\begin{aligned}\hat{g}_{\mu\nu} &= \eta_{\mu\nu} + \epsilon\hat{h}_{\mu\nu} \\ &= \eta_{\mu\nu} + \epsilon(h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu})\end{aligned}$$

i.e.

$$\boxed{\hat{h}_{\mu\nu} = h_{\mu\nu} - 2\xi_{(\mu,\nu)}} \qquad \text{(Gauge Transformation of } h_{\mu\nu})$$

We can check that

$$\hat{R}_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\rho}$$

i.e. the Riemann curvature tensor (and hence the Einstein tensor) are gauge invariant (to first order).

## 8.3 The Newtonian Limit

## 8.4 Gravitational Waves

### 8.4.1 The Linearised Vacuum Solutions

### 8.4.2 Energy Transfer

# 9 Einstein-Maxwell Theory

## 9.1 The Field Equations

The variables of Einstein-Maxwell theory are:

- The metric $g_{\mu\nu}$
- Maxwell tensor $F_{\mu\nu}$
- Stress-energy tensor $T_{\mu\nu}$
- Current 4-vector $J^\mu$

If $\rho$ is the proper electric density and $u^\mu$ the 4-velocity of the charge, then

$$J^\mu = \rho u^\mu$$

At each point $p \in M$, we construct an orthonormal tetrad

$$\{\lambda_{(a)}{}^{\mu}\}_{a=0}^{3}$$

where

$$\lambda^{\mu}_{(0)}\lambda_{(0)\mu} = -1$$

$$\lambda^{\mu}_{(0)}\lambda_{(i)\mu} = -0 \qquad (i = 1, 2, 3)$$

$$\lambda^{\mu}_{(i)}\lambda_{(j)\mu} = \delta_{(i)(j)} \qquad (i, j = 1, 2, 3)$$

Then $\lambda^{\mu}_{(a)}\lambda_{(b)\mu} = \eta_{(a)(b)} = \mathrm{diag}(-1, 1, 1, 1)$. The orthonormal tetrad of the Maxwell tensor are

$$F_{(a)(b)} = F_{\mu\nu}\lambda^{\mu}_{(a)}\lambda^{\nu}_{(b)} = -F_{(b)(a)}$$

These components define the electric and magnetic 3-vectors

$$F_{(a)(b)} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & -B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$\vec{E} = (E_1, E_2, E_3)$
$\vec{B} = (B_1, B_2, B_3)$

Maxwell's equations in flat space are

$$\vec{\nabla} \times \vec{B} - \partial_t \vec{E} = \vec{J}$$

$$\vec{\nabla} \cdot \vec{E} = \rho$$

$$\vec{\nabla} \times \vec{E} - \partial_t \vec{B} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

These may be written in tensor notation as

$$F^{\mu\nu}{}_{,\nu} = J^{\mu} \qquad \text{(i)}$$

$$F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0 \qquad \text{(ii)}$$

where $J^{\mu} = (\rho, \vec{J})$

$$F^{0i} = E^{i} \qquad (i = 1, 2, 3)$$

$$F_{ij} = \epsilon^{ijk} B_k \qquad (i, j, k = 1, 2, 3)$$

The covariant generalisations of these equations are obtained by the "comma goes to semi-colon" rule:

$$\boxed{\begin{aligned} &F^{\mu\nu}{}_{;\nu} = J^{\mu} \\ &F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0 \end{aligned}} \qquad \text{(Maxwell's equations in curved spacetime)}$$

It is easy to prove that the second equation above is equivalent to (ii) and hence there exists a 4-potential $A_\mu$ such that

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} = A_{\mu;\nu} - A_{\nu;\mu}$$

For source-free regions ($J^\mu = 0$) in a vacuum, we still have a contribution to the energy-momentum tensor coming from the Maxwell tensor. The electromagnetic Lagrangian density is defined by

## 9.2 The Reissner-Nordstrom Solution