# List of Theorems from Geometry 2321, 2322

Stephanie Hyland
hylandst@tcd.ie

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*There's already a list of geometry theorems out there, but the course has changed since, so here's a new one. They're in order of 'appearance in my notes', which corresponds reasonably well to chronological order. 24 onwards is Hilary term stuff.*

The following theorems have actually been asked (in either summer or schol papers):
5, 7, 8, 17, 18, 20, 21, 22, 26, 27, 30 a), 31 (associative only), 35, 36 a), 37, 38, 39, 43, 45, 47, 48.

Definitions are also asked, which aren't included here.

1. On a finite-dimensional real vector space, the statement 'V is open in M' is independent of the choice of norm on M.
2. Let $\mathbb{R}^n \supset V \xrightarrow{f} \mathbb{R}^m$ be differentiable with $f = (f^1, ..., f^m)$. Then $\mathbb{R}^n \xrightarrow{f'} \mathbb{R}^m$, and $f' = \left(\frac{\partial f^i}{\partial x_j}\right)$
3. Let $M \supset V \xrightarrow{f} N, a \in V$. Then $f$ is differentiable at $a \Rightarrow f$ is continuous at $a$.
4. $f = (f^1, ...f^n)$ continuous $\Leftrightarrow f^i$ continuous, and same for differentiable.
5. The chain rule for functions on finite-dimensional real vector spaces.
6. The chain rule for functions of several real variables.
7. Let $\mathbb{R}^n \supset V \xrightarrow{f} \mathbb{R}$, $V$ open. Then $f$ is $C^1 \Leftrightarrow \frac{\partial f}{\partial x_i}$ exists and is continuous, for $i = 1, ..., n$.
8. Let $\mathbb{R}^n \supset V \xrightarrow{f} \mathbb{R}$, $V$ open. $f$ is $C^2$. Then $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$
9. $(\phi \circ \psi)_* = \phi_* \circ \psi_*$, where $\phi, \psi$ are maps of manifolds, and $\phi_*$ is the push-forward of $\phi$.
10. Something about the components of the push-forward. Doubt this'll come up.
11. $B_x(a, r)$ is open in $X$.
12. $M \supset X \xrightarrow{f} Y \subset N$, $M, N$ normed, finite-dimensional vector spaces. Then $f$ is continuous at $a \Leftrightarrow$ for each open NBD $V$ of $f(a)$ in $Y$, there exists an open NBD $W$ of $a$ in $X$, such that $f(W) \subset V$.
13. Let $X \subset M$, $M$ a finite-dimensional normed space. Then any union of open sets is open, any finite intersection of open sets is open, and a set $V \subset X$ is open in $X \Leftrightarrow$ there exists $W$ open in $M$ such that $V = W \cap X$.
14. Let $X, Y$ be topological spaces, with $f : X \to Y$. Then $f$ is continuous $\Leftrightarrow V$ open in $Y \Rightarrow f^1 V$ open in $X$.
15. $f, g$ continuous $\Rightarrow f \circ g$ continuous.
16. The mean value theorem for vector-valued functions.

17. The inverse function theorem.
18. The implicit function theorem.
19. Something about coordinate systems, the statement of the theorem is literally 8 times the length of the proof.
20. If $u_i$ is a basis for $M$, then the coordinate functions $u^i$ are the basis for the dual space $M^*$.
21. The pull-back commutes with the differential.
22. The chain rule for maps of manifolds *(identical to theorem 9).*
23. Properties of the operator norm. *(Anachronistic - I didn't have it down as a theorem when I first made up my notes.)*
24. If $\omega$ is an exact one-form on $X$, then $\int \omega$ is path-independent.
25. Integration by parts.
26. Let $u_i$ be a basis for $M$. Then, for example, $u^i \otimes u_j \otimes u^k$ is a basis for $M^* \otimes M \otimes M^*$.
27. Contraction is well-defined independent of choice of basis.
28. The group $S_r$ acts on the vector space $\otimes^r M^*$ by linear transformations.
29. If $T \in \otimes^r M^*$, then $\sum_{\sigma \in S_r} \varepsilon^\sigma \sigma \cdot T$ is skew-symmetric.
30. Let $S \in \otimes^s M^*, T \in \otimes^t M^*$. Then
    a) $a[(aS) \otimes T] = a[S \otimes T] = a[S \otimes (aT)]$
    b) $a[S \otimes T] = (-1)^{st} a[T \otimes S]$
31. The wedge product is bilinear, associative, and super-commutative.
32. Let $u_1, ..., u_n$ be a basis for $M$. And let $i_1 < ... < i_r, j_1 < ... < j_r$. Then $u^{i_1} \wedge ... \wedge u^{i_r}[u_{j_1}, ..., u_{j_r}] = 1$ if $i_k = j_k$, and 0 otherwise.
33. Let $\Lambda^r M^*$ be the vector space of skew-symmetric tensors of type $\underbrace{M \times ... \times M}_{r} \to \mathbb{R}$. Let $u_i, ..., u_n$ be a basis of $M$. Then
    a) $\Lambda^r M^* = \{0\}$ if $r > n$.
    b) The tensor $\sum_{i_1 < ... < i_n} \alpha_{i_1,...,i_r} u^{i_1} \wedge ... \wedge u^{i_r}$ has components $\alpha_{i_1,...,i_r}$ for $i_1 < ... < i_n$.
    c) $\{u^{i_1} \wedge ... \wedge u^{i_r}\}_{i_1 < ... < i_r}$ is a basis for $\Lambda^r M^*$, so its dimension is $\frac{n!}{r!(n-r)!}$
34. $T_*, T^*$ preserve commutative diagams, ie $(UT)_* = U_* T_*$, and $(UT)^* = T^* U^*$.
35. The pullback/pushforward preserve the tensor product, skew-symmetry, and wedge product.
36. If $M$ is an n-dimensional, real, oriented vector space with non-degenerate symmetric scalar product, then
    a) The volume form $u^1 \wedge ... \wedge u^n$ is independent of choice of standard basis $u_1, ..., u_n$ of $M$.
    b) For any positively oriented basis $w_1, ..., w_n$, we have $\text{vol}(w_1, .., .w_n) = \sqrt{|\det(w_i|w_j)|}$.
37. Let $u_1, ..., u_n$ be a standard basis with $(u_i|u_i) = \delta_i = \pm 1$. Then $\star(u^1 \wedge ... \wedge u^r) = \delta_{r+1} ... \delta_n u^{r+1} \wedge ... \wedge u^n$.
38. Defining the linear operator $\Omega^r(V) \xrightarrow{d} \Omega^{r+1}(V)$, and then
    a) The Leibniz rule: $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (1)^r \omega \wedge (d\eta)$ if $\omega$ is an r-form.
    b) $dd\omega = 0$.

c) The definition of $d\omega$ is independent of the choice of coordinates on $V$.

39. The pull-back of differential forms is a linear operator, preserves wedge products, and commutes with the differential.

40. $\int_V \omega$ is independent of choice of coordinates on $V$. *(In my notes, there are quotation marks around this being a theorem, so it may not be examinable.)*

41. Let $X \xrightarrow{\phi} Y$ be a diffeomorphism of n-dimensional oriented manifolds, preserving orientations, and $V$ a coordinate domain on $X, \omega$ an n-form on $Y$, then... $\int_{\phi V} \omega = \int_V \phi^* \omega$.

42. $\int_X \omega$ is independent of the choice of partition of unity. *(Another potentially fake theorem.)*

43. Stokes' Theorem.

44. Solutions to the Laplace equation on a region are unique. *(This is almost certainly not a proper theorem we did, but hey, it might come up in methods.)*

45. Poincarè's Lemma.

46. Let $u_1, ..., u_n$ be a moving frame on a Riemannian n-dimensional manifold. Then there exists a unique skew-symmetric $n \times n$ matrix $\Omega = (\omega^i_j)$ of 1-forms such that $du^i = -\omega^i_j \wedge u^j$.

47. $x_1, ..., x_r$ are linearly independent if any only if $x_1 \wedge ... \wedge x_r \neq 0$. *(This and the next 'theorem' came up in a problem set.)*

48. $x_1, ..., x_r$ generate the same subspace as $y_1 ..., y_r$ if and only if $x_1 \wedge ... \wedge x_r$ is a scalar multiple of $y_1 \wedge ... \wedge y_r$.