# Some Geometry Definitions

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*These seem to be some of the more commonly-asked definitions. My definitions here are by no means perfect; they're pretty much a hybrid of what we were given in class with existing definitions from previous course notes (namely 211 and 221). It's always hard to know how long a definition should be. I might update this with more definitions at some point, in which case the date will change.*

**velocity vector:** Let $\alpha(t)$ be a sequence of real-valued functions (a curve). If $f$ is a real-valued $C^\infty$ function on an open set in $\mathbb{R}^n$ containing $\alpha(t)$, then the rate of change of $f$ along $\alpha$ at $t$ is...

$$\begin{aligned}\frac{df}{dt}(\alpha(t)) &= \frac{df}{dt}(\alpha^1(t), \alpha^2(t), ..., \alpha^n(t)) \\ &= \left[\frac{d}{dt}\alpha^1(t) \cdot \frac{\partial}{\partial x^1_{\alpha(t)}} + ... + \frac{d}{dt}\alpha^n(t) \cdot \frac{\partial}{\partial x^n_{\alpha(t)}}\right] f \\ &= \dot{\alpha}(t) f\end{aligned}$$

$\dot{\alpha}(t)$ is the velocity vector of $\alpha(t)$. Its components are $\frac{d}{dt}\alpha^i(t) = \frac{d}{dt}x^i(\alpha(t))$, ie the rate of change of $x^i$ along $\alpha(t)$.

**tangent space to V at a:** $(TaV)$. The set of linear combinations of $\frac{\partial}{\partial x^1_a}, ..., \frac{\partial}{\partial dx^n_a}$. (And these form a basis of an n-dimensional real vector space, called the tangent space to V at a.)

**tangent bundle:** If $X$ is a manifold, the tangent bundle $TX$ is

$$TX = \bigcup_{x \in X} T_x X$$

**push-forward:** Let $\phi$ be a map of manifolds, $\mathbb{R}^n \supset V \xrightarrow{\phi} W \subset \mathbb{R}^m$. Define the push-forward, $TV \xrightarrow{\phi_*} TW$ as, if $v \in TV$, then

$$[\phi_* v] f = v [f \circ \phi] = v [\phi^* f]$$

for each real-valued differentiable function $f$ on a neighbourhood of $\phi(\alpha(t))$, where $v = \dot{\alpha}(t)$. (And $TV$ is the tangent bundle of $V$).

$$[\phi_* v]f = \frac{d}{dt} f(\phi(\alpha(t))) = (\phi \dot{\circ} \alpha)(t) f$$
$$\therefore \phi_* v \in T_{\phi(\alpha(t))} V$$

Or in other words,

$$\phi_*[\text{velocity vector of } \alpha \text{ at } t] = [\text{velocity vector of } (\phi \circ \alpha) \text{ at } t]$$

**pull-back:** Let $X \xrightarrow{\phi} Y$ be a map of manifolds. For each scalar field $f$ on $Y$ with domain $V$ open in $Y$, define the pull-back;

$$\phi^* f = f \circ \phi$$

a scalar field on $X$ with domain $\phi^{-1}V$ open in $X$.

If $\omega$ is a 1-form on $Y$ with domain $V$, then the pull-back $\phi^*\omega$ is the 1-form on $X$ with domain $\phi^{-1}V$, defined by

$$< (\phi^*\omega)_x, v > = < \omega_{\phi(x)}, \phi_* v >$$

for all $v \in T_x X$.

**push-forward, pull-back (for linear forms):** For each linear operator $M \xrightarrow{T} N$ on finite-dimensional real vector spaces $M, N$, and each integer $r \geq 1$, we define the push-forward $\otimes^r M \xrightarrow{T_*} \otimes^r N$, and pull-back $\otimes^r M^* \xleftarrow{T^*} \otimes^r N^*$ as follows;

$r = 1$

$$\otimes^1 M = M \xrightarrow{T_*} N = \otimes^1 N, \quad T_* = T$$
$$\otimes^1 M^* = M^* \xleftarrow{T^*} N^* = \otimes^1 N^*, \quad < T^* f, x > = < f, Tx > = < f, T_* x >$$

For all $f \in N^*, x \in M$.

$r \geq 1$

$$(T_* S)(f_1, ..., f_r) = S(T^* f_1, ..., T^* f_r)$$
$$(T^* S)(x_1, ..., x_r) = S(T_* x_1, ..., T_* x_r) = S(T x_1, ..., T x_r)$$

**n-dimensional coordinate system on a topological space $X$ with domain $U$:** is a homeomorphism $y = (y^1, y^2, ..., y^n)$ of an open set $U$ in $X$ onto an open set $y(U)$ in $\mathbb{R}^n$, with the usual topology in $\mathbb{R}^n$.

**$C^r$-compatible:** Let $y = (y^1, ..., y^n)$ with domain $V$, and $z = (z^1, ..., z^n)$ with domain $W$ be two coordinate systems on $X$. They are $C^r$-compatible if each $y^i$ is a $C^r$ function of $z^1, ..., z^n$, and each $z^i$ is a $C^r$ functio of $y^1, ..., y^n$ on $V \cap W$. They are $C^\infty$ compatible if they are $C^r$-compatible for all $r$.

**smooth n-dimensional manifold:** A topological space $X$ is such if a collection (or atlas) of mutually $C^\infty$-compatible coordinate systems is given on $X$ whose domain covers $X$.

**differential of $f$ at $a$:** If $f$ is a scalar field at $a \in X$, $X$ a manifold, then the differential of $f$ at $a$ is the linear functional

$$df_a : T_a X \to \mathbb{R}$$

whose value, $< df_a, v >$ on $v \in TaX$ is $vf$.

$$< df_a, v >= vf = \text{rate of change of } f \text{ along } v$$

$df_a$ has components $< df_a, \frac{\partial}{\partial y_a^j} >= \frac{\partial f}{\partial y^j}(a)$.

$$\therefore df_a = \frac{\partial f}{\partial y^j}(a) dy_a^j$$

**vector field:** $v$, with domain $V$, is a function, $\forall a \in V$

$$a \longmapsto v_a \in T_a X$$

Where $X$ is an n-dimensional manifold, $V$ an open subset of $X$.

**integral of $\omega$ over $\alpha$ (1-form):** If $\omega$ is a 1-form on a manifold $X$, and $[t_1, t_2] \xrightarrow{\alpha} X$, $t \mapsto \alpha(t)$, is a parametrised path in $X$, then the integral of $\omega$ over $\alpha$ is

$$\int_\alpha \omega = \int_{t_1}^{t_2} < \omega_{\alpha(t)}, \dot{\alpha}(t) > dt$$

Thus if $f, g$ are scalar fields on $X$, then

$$\int_\alpha f dg = \int_{t_1}^{t_2} \left[ f(\alpha(t)) \cdot \frac{d}{dt} g(\alpha(t)) \right] dt$$

**integral of $\omega$ (n-form):** Let $\omega$ be an n-form on an oriented n-dimensional manifold $X$, and $V$ be the domain of positively orietned coordinates $y^1, ..., y^n$. $\omega = f(y^1, ..., y^n) dy^1 \wedge ... \wedge dy^n$, say. Then

$$\int_V \omega = \int_{y(V)} f(x_1, ..., x_n) dx_1 ... dx_n$$

**differential r-form:** Let $X$ be a smooth n-dimensional manifold, and $V$ an open subset. Let $r \geq 1$. Then a differential r-form $\omega$ with domain $V$ is a function on $V$;

$$a \longmapsto \omega_a \in \Lambda^r (T_a X)^*$$

If $y_i$ are coordinates on $X$ with domain $V$, for each $a \in V$, $dy_a^1, ..., dy_a^n$ are a basis for $(T_a X)^*$

$$\therefore \omega_a = \sum_{i_1 < ... < i_r} \omega_{i_1, ..., i_r}(a) dy_a^{i_1} \wedge ... \wedge dy_a^{i_r}$$

Where $\omega_{i_1, ..., i_r}$ are $C^\infty$ scalar-valued functions on $V$, called the components of $\omega$ with respect to the coordinates $y^i$.