# Rough notes for Maths 543

Please send corrections and comments to Conor Houghton: houghton@maths.tcd.ie

## Lecture 9

At the last Lecture we introduced the exterior derivative[1]

$$d : \Omega^p \to \Omega^{p+1}. \tag{1}$$

We noted that $d^2 = 0$ and so

$$\ldots \xrightarrow{d} \Omega^{p-1} \xrightarrow{d} \Omega^p \xrightarrow{d} \Omega^{p+1} \xrightarrow{d} \ldots \tag{2}$$

is a complex[2]

. A element of $\ker d$ is called a **closed form**, thus, $\omega$ is closed if $d\omega$ is zero. The terminology is taken from the definition of a closed chain. An **exact form** is an element of $\mathrm{im}\, d$; $\omega \in \Omega^p$ is exact if $\omega = d\eta$ for some $\eta \in \Omega^{p-1}$.

The $p$-**cohomology** group is the group generated over $\mathbf{R}$ by the equivalence classes of closed $p$-forms, where the equivalence is given by addition of an exact $p$-form:

$$H^p(M, \mathbf{R}) = \ker d / \mathrm{im}\, d. \tag{3}$$

It might seem at first sight that all closed forms are exact but that isn't true. If it was, all cohomology groups would be trivial. We'll look briefly at two simple examples.

Consider the punctured plane $\mathbf{R}^2 \setminus \{(0,0)\}$ and the one-form:

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy. \tag{4}$$

This one-form is closed,

$$d\omega = \left[-\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) + \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)\right] dx \wedge dy. \tag{5}$$

Locally, it is exact

$$\omega = d\theta \tag{6}$$

where

$$\theta = \tan^{-1}\frac{y}{x} \tag{7}$$

---

[1]It is common to write $\Omega^p$ as a shorthand for $\Omega^p M$.

[2]I should have defined a **complex** before, a sequence of maps

$$\ldots \xrightarrow{\phi} A \xrightarrow{\phi} B \xrightarrow{\phi} C \xrightarrow{\phi} \ldots$$

is a complex if $\mathrm{im}\,\phi \leq \ker\phi$ where obviously I have abused notation by giving all the maps the same name.

but $\theta$ is not a function on $\mathbf{R}^2 \setminus \{(0,0)\}$; it is only single valued on a cut plane, $\mathbf{R}^2 \setminus \{$the $x_1$-axis$\}$ for example. Furthermore, if there is some other $\eta$ such that $\omega = d\eta$ then $d\eta = d\theta$ and this means that $\eta$ and $\theta$ only differ by a constant. We can see from this that $\omega$ generates a non-trivial class in $H^1(\mathbf{R}^2 \setminus \{(0,0)\})$. In fact, $H^1(\mathbf{R}^2 \setminus \{(0,0)\}) = \mathbf{R}$.

The second simple example is the two-torus. As we discussed, a model for the torus is a rectangle in the plane with opposite edges identified.[3] Now, $dx_1$ and $dx_2$ are closed one-forms on the torus. Neither is exact since, with the identifications, neither $x_1$ nor $x_2$ is single valued on the parallelogram. In fact, it turns out that $H^1(T^2) = \mathbf{R} \oplus \mathbf{R}$.

In the case of the punctured plane there is a closed form that is not exacted. There is also a non-contractible loop. In the case of the torus, there are two non-cohomologous closed forms, there are also two non-homologous cycles. These are examples of a more general relationship between homology and cohomology: they are dual to each other. To study this duality we first need to discuss integration on manifolds.[4]

# Integration on manifolds

We begin with a **standard $p$-simplex**, $\sigma_p$. It is the simplex with vertices

$$\begin{aligned} x_0 &= (0,0,\ldots,0) \\ x_1 &= (1,0,\ldots,0) \\ &\vdots \\ x_p &= (0,0,\ldots,1). \end{aligned} \tag{8}$$

On the simplex any $p$-form can be written

$$\omega = f dx_1 \wedge \ldots dx_p. \tag{9}$$

We define

$$\int_{\sigma_p} \omega = \int_{\sigma_p} f dx_1 dx_2 \ldots dx_p \tag{10}$$

In other words, it is easy to define integration on a standard simplex because there are coordinates.

A singular $p$-simplex is a map

$$\lambda : \sigma_p \to M \tag{11}$$

We can define the integral of a $p$-form $\omega \in \Omega^p M$ over a $p$-simplex in $M$ by the pullback

$$\int_\lambda \omega = \int_{\sigma_p} \lambda^* \omega. \tag{12}$$

[3]This is for real manifolds, in fact, there is only one torus up to diffeomorphisms, it can be modeled by a square with identifications. However for complex manifold there are more tori, since holomorphism is a more restrictive equivalence.

[4]Of course, we are doing all of this backwards, the more elegant approach is to observe that the homology permits a dualization: the cohomology and then demonstrate that the de Rham cohomology realized this dualization for manifolds.

The integral is invariant under reparameterization of the standard simplex. If $p : \sigma_p \rightarrow \sigma_p$ is onto then it follows from the fact that a $p$-form changes by a Jacobian factor under change of variables that

$$\int_\lambda \omega = \int_{\lambda \circ p} \omega \tag{13}$$

We can generalize this definition to a $p$-cycle. Recall that the singular homology is derived from singular $p$-cycles

$$c = \sum_i a_i \lambda_i \tag{14}$$

where $a_i \in \mathbf{R}$ and $\lambda_i$ a singular p-simplex. Now

$$\int_c \omega = \sum a_i \int_{\sigma_p} \lambda_i^* \omega. \tag{15}$$

## Stoke's theorem

Thus, we can define integration and this maps a $p$-form and a $p$-cycle to a real number. This inner-product is well-defined on homology and cohomology because of **Stoke's theorem**.[5] This states that

$$\int_c d\omega = \int_{\partial c} \omega \tag{16}$$

It is easy to prove Stoke's theorem if you prove the important result that, for $f : M \rightarrow N$

$$df^* \omega = f^* d\omega \tag{17}$$

This follows by direct computation inside each coordinate patch, if[6]

$$\omega = \omega_{i \ldots k} dx_i \wedge \ldots \wedge dx_k \tag{18}$$

[5] Stokes proved his theorem in a more specific context. Stoke, incidently, was Irish, born in Skreen, Co Sligo in 1819. He spent most of his life in Cambridge, England, where was was, among other thing, Lucasian professor.

[6] For some reason I feel this is a good moment to admit that there is a reasonably common though unevenly applied convention where you write $dx^i$ and regard the index in

$$\frac{\partial}{\partial x^i}$$

as begin a subscripted index and only summing between up and down indices. Thus, a one-form is written

$$\omega = \omega_i dx^i$$

and a vector as

$$V = V^i \frac{\partial}{\partial x^i}.$$

so that

$$\omega(V) = \omega_i V^i.$$

For some reason I have never done this myself and haven't done it here, but it is a useful thing.

then, if $(y_1, \ldots, y_k) = f(x_1, \ldots, x_k)$

$$f^*\omega = \omega_{i\ldots k} J^{p-1}_{i\ldots ki'\ldots k'} dy_{i'} \wedge \wedge \ldots \wedge dy_{k'} \tag{19}$$

where

$$J^{p-1}_{i\ldots ki'\ldots k'} = \frac{\partial x_i}{\partial y_{i'}} \cdots \frac{\partial x_k}{\partial y_{k'}} \tag{20}$$

is the Jacobian for $p-1$ variables. Taking the exterior derivative

$$\begin{aligned} df^*\omega &= \frac{\partial}{\partial y_l'}(\omega_{i\ldots k} J^{p-1}_{i\ldots ki'\ldots k'}) dy_l' \wedge dy_{i'} \wedge \ldots \wedge dy_{k'} \\ &= \frac{\partial \omega_{i\ldots k}}{\partial x_l} \frac{\partial x_l}{\partial y_{l'}} J^{p-1}_{i\ldots ki'\ldots k'} dy_l' \wedge dy_{i'} \wedge \ldots \wedge dy_{k'} \end{aligned} \tag{21}$$

Now,

$$\begin{aligned} f^*d\omega &= f^*\left(\frac{\partial \omega_{i\ldots k}}{\partial x_l} dx_l \wedge dx_i \wedge \ldots \wedge dx_k\right) \\ &= \frac{\partial \omega_{i\ldots k}}{\partial x_l} J^p_{li\ldots jl'i'\ldots j'} dy_{l'} \wedge dy_{i'} \wedge \ldots \wedge dy_k' \end{aligned} \tag{22}$$

And the equality follows.

To return to the proof of Stoke's theorem, we can take the cycle to be a single singular simplex $\lambda$. We have

$$\int_\lambda d\omega = \int_{\sigma_p} \lambda^*(d\omega) = \int_{\sigma_p} d(\lambda^*\omega) \tag{23}$$

on one side and

$$\int_{\partial\lambda} \omega = \int_{\partial\sigma_p} \lambda^*(\omega) \tag{24}$$

on the other, so we only need to prove that

$$\int_{\partial\sigma_p} \eta = \int_{\sigma_p} d\eta \tag{25}$$

for any exact form $d\eta \in \Omega^p\sigma_p$. However, this is easy, let

$$\eta = \sum_i h_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge dx_p \tag{26}$$

so that

$$d\eta = \left(\sum_i (-)^i \frac{\partial h_i}{\partial x_i}\right) dx_1 \wedge \ldots \wedge dx_p \tag{27}$$

and then evaluate the integral over $\sigma_p$. In other words, the theorem reduces to the classical version of the Stoke's theorem.

### De Rham's theorem

So, say $\omega$ is a closed form and $c$ is a closed cycle, then

$$\int_{c+\partial c'} \omega = \int_c \omega + \int_{c'} d\omega = \int_c \omega \tag{28}$$

and

$$\int_c (\omega + d\omega') = \int_c \omega + \int_{\partial c} \omega' = \int_c \omega. \tag{29}$$

Thus, we have a well-defined inner product

$$\begin{aligned} H^p(M, \mathbf{R}) \times H_p(M, \mathbf{R}) &\to \mathbf{R} \\ ([\omega], [c]) &\mapsto \int_c \omega \end{aligned} \tag{30}$$

and, with respect to this inner product, $d$ and $\partial$ are adjoint operators. de Rham's theorem states that for $M$ compact this inner product is non-degenerate. In other words, if

$$\int_c \omega = 0 \tag{31}$$

for all cycles $c$ then $\omega$ is exact. If it holds for all forms $\omega$ then $c$ is a boundary.

Recall that the Betti numbers were defined as the dimensions of the homology groups

$$b_r = \dim H_r(M, \mathbf{R}) \tag{32}$$

One immediate consequence of de Rham's theorem is that the dual Betti numbers are the same

$$b^r = \dim H^r(M, \mathbf{R}) = b_r \tag{33}$$

So we can calculate the Euler characteristic

$$\chi(M) = \sum (-)^r b^r \tag{34}$$

from analytic data.

Another consequence of de Rham's theorem is **Poincaré's lemma** guaranteeing local exactness. Poincaré's lemma states that a closed form on a contractible neighbourhood is exact on that neighbourhood.

## Properties of the de Rham cohomology groups.

We have described the integration of $p$-forms over $p$-cycles. The basic idea was that the $p$-form on the cycle can be pulled back to a function multiplying a $p$-form on a standard $p$-cycle. This basic idea also gives a definition of the integral of a $n$-form over an **orientable** $n$-dimensional manifold. This integral is used to define a duality between different cohomology groups called the Poincaré duality.

## Integration over a manifold again

We are now going to go through integration on a manifold again. The details will be different since this new definition is designed to apply in different circumstances. However, it should be clear that the basic idea is the same and the two definitions are consistent.

A manifold is called orientable if it is covered by a coordinate neighbourhoods such that, if $U_i \cap U_j$ is not empty, there are coordinates $(x_1, \ldots, x_n)$ on $U_i$ and $(y_1, \ldots, y_n)$ on $U_j$ with

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_n)} > 0. \tag{35}$$

The obvious spaces fail to be orientable: the Möbius strip and real projective space.[7] We can define the integral over the whole manifold by pulling back the integration over different coordinate patches using a **partition of unity**. Thus, assuming the space is compact[8] it can be covered by a finite number of coordinate patches $U_i$. A partition of unity for the cover $\{U_i\}$ is a family of differentiable functions $\epsilon_i$ with $0 \leq \epsilon_(x) \leq 1$, so that $\epsilon_i(x) = 0$ when $x \notin U_i$ and $\epsilon_1(x) + \epsilon_2(x) + \ldots = 1$ for all $x$.[9]

Now if $\omega$ is an $n$-form it can be written

$$\omega = f_i dx_1 \wedge \ldots \wedge dx_n \tag{36}$$

on a coordinate patch $U_i$ with coordinates $(x_1, \ldots, x_n)$.[10] then

$$\int_M \omega = \sum \int_{\phi_i(U_i)} \epsilon_i f_i dx_1 \ldots dx_n \tag{37}$$

where each of the integrations is over some subspace of $\mathbf{R}^n$ and so it is well defined. The sum is converges since it is finite.

The form is pulled back to a coordinate neighbourhood so that it can be integrated. Since the manifold has many overlapping coordinate neighbourhoods, the partition of unity must be used. It can be shown that this definition is independent of the coordinatization and of the partition of unity.

[7] On an orientable space there exist a nowhere vanishing $n$-form $\omega$. Such a form is called **volume form**. Using a volume form you can extend the definition of integration that follows so that it applies to functions, that is we integrate $f$ by integrating $f$ times the volume form.

[8] You can get away with para-compactness, this is the weaker property that there is a cover so that any point is covered by only a finite number of patches.

[9] Partitions of unity are made out of the infinitely differentiable function

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-1/x} & x \geq 0 \end{cases}$$

All the derivatives of this function vanish at the origin so it smoothly interpolates between a function that is constant and one that isn't.

[10] Of course this means that $\phi^* \omega = f dx_1 \wedge \ldots \wedge dx_n$ on $\phi_i(U)$ where $\phi_i$ is the coordinate map.

## Poincaré duality

With all of that out of the way, we have an inner product

$$\begin{aligned} H^r(M) \times H^{n-r}(M) &\rightarrow \mathbf{R} \\ ([\omega], [\eta]) &\mapsto \int_M \omega \wedge \eta. \end{aligned} \tag{38}$$

This inner product is non-degenerate and so there is a duality between $H^r(M)$ and $H^{n-r}(M)$. This means $b^r = b^{n-r}$. One corollary of this is that compact connected orientable manifolds of odd dimension have zero Euler characteristic.

## The cohomology ring

We define the wedge product of two cohomology classes in the obvious way: $[\omega] \wedge [\eta] = [\omega \wedge \eta]$. It is easy to see that this is well-defined and gives the direct sum

$$H^*(M) = \oplus_r H^r(M) \tag{39}$$

a ring structure.