Rough notes for Maths 543

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Lecture 8

We would now like to study cohomology. Cohomology groups are dual to homology groups and have many similar properties. The obvious difference is that the homomorphism in the long exact sequences go in the other direction. However, it turns out that cohomology is more powerful than homology because it is possible to multiply two cohomology classes. This gives the cohomology groups of a space a ring structure. Furthermore, the operator used to construct cohomology groups, the exterior derivative, is a local operator whereas the boundary operator which was used to construct homology is a global one.

Before considering cohomology groups we define vectors, forms and exterior derivatives. The de Rham cohomology will then be introducted in the next lecture.

Vectors and One-forms.

A real function, f, on a manifold, M, is a map

$$f : M \to \mathbf{R} \tag{1}$$

If M is a differentiable manifold, it is easy to define a smooth function. The space of smooth functions over M will be denoted $\mathcal{F}M$.[1] In the case of a function on $\mathbf{R}^3$, it is useful to consider the directional derivative $\mathbf{v} \cdot \nabla$ giving the derivative of the function in the direction of some vector $\mathbf{v}$. On a general manifold, it is not possible to define a vector by say it is a directed line between one point and another.[2] Instead, **vectors** at a point in a manifold will be identified with directional derivatives at that point.

A curve is a map from an open line segment into a manifold

$$c : (a, b) \to M \tag{2}$$

If $c(t)$ is a curve through a point $x = c(t_0)$ then the derivative of a smooth function f at that point is

$$\left.\frac{df}{dt}\right|_{t_0} = \lim_{\delta t \to 0} \frac{f(c(t_0 + \delta t)) - f(c(t_0))}{\delta t} \tag{3}$$

We know that this is the directional derivative of f in the direction of the tangent to c at $t = t_0$. The idea is to identify the tangent to c with this directional derivative. In other words, the tangent vector to c at x is V with

$$V(f) = \frac{df}{dt} \tag{4}$$

[1]This is not a universal notation, in fact, we will change between this notation and another one though the course of this lecture.

[2]We want to define vectors at a point

If $(x_1, x_2, \ldots, x_n)$ are coordinates at x then

$$V(f) = \frac{df}{dt} = \frac{dx_i}{dt}\frac{\partial f}{\partial x_i} \tag{5}$$

where we sum on i and where the curve is given by $(x_1(t), x_2(t), \ldots, x_n(t))$ in the coordinate neighbourhood of of x. Thus, we can write

$$V = V_i \frac{\partial}{\partial x_i} \tag{6}$$

where $V_i = dx_i/dt$.

In other words, a basis for the tangent vectors at the point x is given by[3]

$$(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}) \tag{7}$$

Now, if we change coordinates from $(x_1, x_2, \ldots, x_n)$ to $(y_1, y_2, \ldots, y_n)$ the components of the vector change

$$V = V_i \frac{\partial}{\partial x_i} = V_i \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j} \tag{8}$$

as the components of a contravariant vector. This shows that our definition of a vector coincides with the component bases definition sometimes used on $\mathbf{R}^n$.[4]

The vector space of tangent vectors at a point x is often write T_xM and called the **tangent space** of M at p. A **tangent vector field** is a smooth assignation of a tangent vector at each point in a manifold.[5] The space of tangent vector fields is often denoted $\mathcal{X}M$.

Since T_xM is a vector space, we can define its dual space T_x^*M with $\omega \in T_x^*M$ a map

$$\omega : T_xM \to \mathbf{R}. \tag{9}$$

Of course, since T_xM is an n-dimensional vector space, so is T_x^*M. However, the tangent space isn't just an assignation of a vector space to each point in space, we know how a

[3]Of course, this makes a lot of sense because the coordinate map maps from an open neighbourhood of x to a subspace of $\mathbf{R}^n$ and we can define a curve by taking the inverse image of straight line. By taking the straight lines parallel to the Cartesian axes we get the basis vectors.

[4]The basis $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})$ is sometimes called a **coordinate** basis because it is derived from a coordinate system. Another basis $A_{ij}\frac{\partial}{\partial x_j}$ with $A \in \mathrm{GL}_n\mathbf{R}$ may not arise as a set of tangents to coordinate curves.

[5]It should be intuitively clear what this is supposed to mean. Roughly speaking, near a point a manifold looks like a subset U of $\mathbf{R}^n$. Since the tangent space also looks like $\mathbf{R}^n$ the tangent vector field is a map from U to $\mathbf{R}^n$ and this map is required to be smooth. In fact, later on we will see that a tangent vector field is a smooth section of the tangent bundle TM. One convenient way of defining a smooth vector field without first defining the tangent bundle is to require that the function $V(f)$ is smooth for any smooth $f \in \mathcal{F}M$

vector acts on a function. In fact, if $f \in \mathcal{F}X$ and $V \in T_xM$ then $V(f) \in \mathbf{R}$. Thus, a function defines an element of T_x^*M which we call df:

$$df : V \mapsto V(f) \tag{10}$$

or, put another way, $df(V) = V(f)$.

The thing is, f need not be globally define. On a coordinate patch U there is a smooth map $\phi : U \to \mathbf{R}^n$. Restricting to one of the n coordinates gives a locally defined function and hence, a locally defined element of T_m^*M. In fact, not only is $(dx_1, dx_2, \ldots, dx_n)$ a coordinate basis for T_m^*M, but, since

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial x_i}{\partial x_j} = \delta_{ij} \tag{11}$$

it is the dual basis under the natural inner product

$$\begin{aligned} T_m^*M \times T_mM &\to \mathbf{R} \\ (\omega, V) &\mapsto \omega(V). \end{aligned} \tag{12}$$

The notation is partly justified by noting that the coordinate independence of the contraction $\omega(V)$ implies[6]

$$\omega = \omega_i dx_i = \omega_i \frac{\partial x_i}{\partial y_j} dy_j. \tag{13}$$

An element of the cotangent space T_x^*M is sometimes called a **cotangent**. However, it is more common to refer to them as **one-forms**. We will see why in the next section. We will define what are called p-forms and it will turn out that a p-form with $p = 1$ is a cotangent. Before going on to p-forms, we will look at an important map between different cotangent spaces: the pullback.

A map between two manifolds induces a map between their tangent bundles. For the moment we are restricting our attention to the tangent space at a point, so we note that, given a smooth map $f : M^m \to N^n$ there is a map

$$f_* : T_mM \to T_{f(m)}N \tag{14}$$

defined by

$$(f_*V)(g) = V(g \circ f) \tag{15}$$

In other words, if g is a function on N then $g \circ f$ is a function on M and so we can evaluate $V(g \circ f)$. If m has coordinates $(x_1, x_2, \ldots, x_n)$ this means

$$V = V_i \frac{\partial}{\partial x_i} \tag{16}$$

[6] Of course, there is no real need to justify the notation, after a while and especially after we have considered integration, it will seem very natural. In fact, dx_i is a modern formulation of the classical idea of a small increment in x_i.

and if $f(x_1, x_2, \ldots, x_m) = (y_1, y_2, \ldots, y_n)$ then

$$f_* V = (f_* V)_i \frac{\partial}{\partial y_i} \tag{17}$$

Taking $g = y_j$

$$(f_* V)_j = V_i \frac{\partial y_j}{\partial x_i}. \tag{18}$$

p-forms.

We will start this section by defining a tensor. We'll then go on to discuss p-forms: p-forms are an important class of tensors.

A **tensor**, T, of type (q, r) is a multilinear map which takes q cotangents and r vectors and maps them to the real numbers. At a point x

$$T : (\otimes^q T_x^* M) \otimes (\otimes^r T_x M) \to \mathbf{R} \tag{19}$$

Thus, if $\omega_1, \ldots \omega_q$ are q cotangents and $V_1, \ldots, V_r$ are r vectors

$$T : (\omega_1, \ldots, \omega_q, V_1, \ldots, V_r) \mapsto T(\omega_1, \ldots, \omega_q, V_1, \ldots, V_r) \in \mathbf{R} \tag{20}$$

In this notation, a cotangent is a type (0,1) tensor because it maps a vector to $\mathbf{R}$. In the same way a vector is a type (1,0) tensor.[7]

Furthermore, we can write a tensor in terms of components

$$T = T_{i \ldots jk \ldots l} \frac{\partial}{\partial x_i} \otimes \ldots \otimes \frac{\partial}{\partial x_j} \otimes dx_k \otimes \ldots \otimes dx_l \tag{21}$$

and, under a change of variables the components of a type (q, r) tensor transforms as you might expect. Finally, **a tensor field** is defined in the obvious way.

A p-**form** is a totally antisymmetric tensor of type $(0, p)$. Thus, if ω is a p-form at x

$$\begin{aligned} \omega : \otimes^p T_x M &\to \mathbf{R} \\ (V_1, V_2, \ldots, V_p) &\mapsto \omega(V_1, V_2, \ldots, V_p) \in \mathbf{R} \end{aligned} \tag{22}$$

such that

$$\omega(V_i, V_j, \ldots, V_k) = \text{sign} \begin{pmatrix} i & j & \ldots & k \\ 1 & 2 & \ldots & p \end{pmatrix} \omega(V_1, V_2, \ldots, V_p) \tag{23}$$

[7] The notation is very bad here and it isn't really worth sorting it out since it is usually obvious what is meant. The problem is we have defined a vector as a differential operator on a function: $V : f \mapsto V(f)$. We then defined the cotangent, or one-form, as a vector in the dual space, so $T^* M \ni \omega : V \mapsto \omega(V) \in \mathbf{R}$. Furthermore, we noted that a function f defines a one-form, df where $df : V \mapsto df(V) = V(f)$. However, the dual of the dual of a finite-dimensional vector space is itself, so V can also be thought of as acting on the space of cotangents, $V : \omega \mapsto V(\omega) = \omega(V)$. This is precisely the attitudes taken when pointing out that a vector is a type (1,0) tensor, the problem is you end up with $V(df) = V(f)$, the point being that the action denoted by the brackets is different in each case. Ah well.

So, for example, if ω is a two-form and V_1 and V_2 are both vectors then $\omega(V_1, V_2)$ is a real number with $\omega(V_1, V_2) = -\omega(V_2, V_1)$.

The space of p-forms at x is commonly called $\Omega_x^p M$. Since the anti-symmetry holds trivially for maps with one argument $\Omega_x^1 M = T_x^* M$. The space of p-form fields is called $\Omega^p M$.

A basis for $\Omega_x^p M$ can be constructed from the basis for $T_x^* M$ using the totally antisymmetric product of the basis one-forms. This totally antisymmetric product is called the **wedge product**. The wedge product of one-forms $\{dx_1, dx_2, \ldots, dx_p\}$ is

$$dx_1 \wedge dx_2 \wedge \ldots \wedge dx_p = \sum_{S_p} \text{sign} \begin{pmatrix} i & j & \ldots & k \\ 1 & 2 & \ldots & p \end{pmatrix} dx_1 \otimes dx_2 \otimes \ldots \otimes dx_p \tag{24}$$

where the sum is over all permutations. Thus, for example,

$$\begin{aligned} dx_1 \wedge dx_2 \wedge dx_3 \quad &= dx_1 \otimes dx_2 \otimes dx_3 + dx_2 \otimes dx_3 \otimes dx_1 + dx_3 \otimes dx_1 \otimes dx_2 \\ &-dx_1 \otimes dx_3 \otimes dx_2 - dx_2 \otimes dx_1 \otimes dx_3 - dx_3 \otimes dx_2 \otimes dx_1. \end{aligned} \tag{25}$$

Now, if $(dx_1, dx_2, \ldots, dx_n)$ is a basis for $T_x^* M$ then all the p-forms constructed by wedging together p of these one-forms gives a basis for $\Omega_x^p M$. Because there is an independent p-form of this type for every selection of p different one-forms from the set $\{dx_1, dx_2, \ldots, dx_n\}$,[8] it follows that

$$\text{dim}\Omega_x^p M = \begin{pmatrix} n \\ p \end{pmatrix} \tag{26}$$

For convenience, we define $\Omega_x^0 M = \mathbf{R}$.[9] Note that $\Omega_x^0 M = \mathbf{R}$ as well: all n-forms are proportional to $dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$.

Anyway, this means that a p-form is given locally as

$$\omega = \frac{1}{p!} \omega_{ij\ldots k} dx_i \wedge dx_j \wedge \ldots \wedge dx_k \tag{27}$$

The factorial prefactor is included so that the components $\omega_{ij\ldots k}$ are also the components of the corresponding tensor as given above. We could get rid of it by only including each basis vector once

$$\omega = \sum_{i<j<\ldots<k} \omega_{ij\ldots k} dx_i \wedge dx_j \wedge \ldots \wedge dx_k. \tag{28}$$

If $(y_1, y_2, \ldots, y_n)$ is another basis the antisymmetry means that

$$\omega = \frac{1}{p!} \omega_{ij\ldots k} dx_i \wedge dx_j \wedge \ldots \wedge dx_k = \frac{1}{p!} \omega_{ij\ldots k} \frac{\partial(x_i, x_j, \ldots, x_k)}{\partial(y_i, y_j, \ldots, y_k)} dy_i \wedge dy_j \wedge \ldots \wedge dy_k \tag{29}$$

[8] Obviously $dx_i \wedge dx_j \wedge \ldots \wedge dx_k = 0$ is some one-form dx_l appears twice. If $dx_i \wedge \ldots \wedge dx_j$ and $dx_k \wedge \ldots \wedge dx_l$ contain the same one-forms but in a different order, then they are either equal or one is equal to minus the other, depending on weather the reordering is even or odd.

[9] This means that a function is a zero-form field, that is, $\mathcal{F}M = \Omega^0 M$.

Of course, the definition of the wedge product need not be restricted to basis one forms, we can define the wedge (or **exterior**) product of two forms in a basis independent way: if $\zeta \in \Omega^q_x M$ and $\eta \in \Omega^r_x M$ then $\zeta \wedge \eta \in \Omega^{q+r}_x M$ is defined by[10]

$$(\zeta \wedge \eta)(V_1, V_2, \ldots, V_{q+r}) = \frac{1}{q!r!} \sum_{S_{q+r}} \text{sign} \begin{pmatrix} i & \ldots & l \\ 1 & \ldots & q+r \end{pmatrix} \zeta(V_i, \ldots, V_j)\eta(V_k, \ldots, V_l) \quad (30)$$

The wedge product is associative: $(\zeta \wedge \eta) \wedge \xi = \zeta \wedge (\eta \wedge \xi)$. It has graded commutativity properties, if $\zeta \in \Omega^q M$ and $\eta \in \Omega^r M$ then $\zeta \wedge \eta = (-)^{qr} \eta \wedge \zeta$. It follows that $\zeta \wedge \zeta = 0$ if q is odd.

Thus, there is a graded algebra of forms

$$\Omega^*_x M = \Omega^0_x M \oplus \Omega^1_x M \oplus \ldots \oplus \Omega^n_p M \quad (31)$$

with multiplication given by the wedge product.

Notice that a vector V defines a map $\Omega^p_x M \to \Omega^{p-1}_x M$ by partial evaluation: $V : \omega \mapsto i_V \omega$ where $i_V \omega(V_1, \ldots, V_{p-1}) = \omega(V, V_1, \ldots, V_{p-1})$. This map is called the **interior product**.

The exterior derivative

The exterior derivative is a differential operator which maps p-forms to $(p+1)$-forms. It is usually defined in terms of its action on components, once we have done that and calculated some of its properties, we will see that these properties define the action of the exterior derivative. In other words, the component based definition is actually basis independent.

Given $\omega \in \Omega^p M$, in a coordinate neighbourhood

$$\omega = \frac{1}{p!} \omega_{ij\ldots k} dx_i \wedge dx_j \wedge \ldots \wedge dx_k. \quad (32)$$

and we define the **exterior derivative** $d : \Omega^p \to \Omega^{p+1}$ by

$$d\omega = \frac{1}{p!} \frac{\partial \omega_{ij\ldots k}}{\partial x_l} dx_l \wedge dx_i \wedge dx_j \wedge \ldots \wedge dx_k. \quad (33)$$

Thus, if $f \in \Omega^0 M = \mathcal{F} M$ then

$$df = \frac{df}{dx_i} dx_i. \quad (34)$$

We can see from this that the exterior derivative of a function is df, you could think that this justified the notation used for df, it is better though to think of the exterior derivative as generalizing the construction of df from f to general p and $p+1$ forms.

[10] The factorial factors are convenient, but note, this isn't a universal convention, most people put them in, some people don't, lots of people can never quite remember weather they do or not.

As another example, consider $M = \mathbf{R}^3$. An element of $\mathcal{F}\mathbf{R}^3$ is just an ordinary function in space. $\Omega^1_x\mathbf{R}^3$ is three dimensional and so we can identify a one-form $\zeta \in \Omega^1_x\mathbf{R}^3$ with a 3-vector at that point

$$\zeta = \zeta_1 dx_1 + \zeta_2 dx_2 + \zeta_3 dx_3 \leftrightarrow \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} \tag{35}$$

With this identification d acts on functions over space as grad:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x3} dx_3 \tag{36}$$

Furthermore, $\Omega^2_x\mathbf{R}^3$ is also three dimensional and we can also identify two-forms with vectors[11]

$$\eta = \eta_1 dx_2 \wedge dx_3 + \eta_2 dx_3 \wedge dx_1 + \eta_3 dx_1 \wedge dx_2 \leftrightarrow \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} \tag{37}$$

If ζ is a one-form with the components above

$$d\zeta = \left(\frac{\partial \zeta_3}{\partial x_2} - \frac{\partial \zeta_2}{\partial x_3} \right) dx_2 \wedge dx_3 + \left(\frac{\partial \zeta_1}{\partial x_3} - \frac{\partial \zeta_3}{\partial x_1} \right) dx_3 \wedge dx_1 + \left(\frac{\partial \zeta_2}{\partial x_1} - \frac{\partial \zeta_1}{\partial x_2} \right) dx_1 \wedge dx_2 \tag{38}$$

and so it is the curl. Finally, $\Omega^2_x\mathbf{R}^3$ is one-dimensional and elements are of the form $f dx_1 \wedge dx_2 \wedge dx_3$. They can be identified with functions in the obvious way. If η is a two-form with the components given above, then

$$d\eta = \left(\frac{\partial \eta_1}{\partial x_1} + \frac{\partial \eta_2}{\partial x_2} + \frac{\partial \eta_3}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3 \tag{39}$$

and this is div.

Thus, d is defined as a map $d : \Omega^p M \to \Omega^{p+1} M$ but in the case of $\mathbf{R}^3$ both the one-form fields and two-form fields can be identified with vector fields and, if this is done, the exterior derivative gives us the sequence of maps

$$0 \longrightarrow \Omega^0\mathbf{R}^3 \stackrel{\text{grad}}{\longrightarrow} \Omega^1\mathbf{R}^3 \stackrel{\text{curl}}{\longrightarrow} \Omega^2\mathbf{R}^3 \stackrel{\text{div}}{\longrightarrow} \Omega^3\mathbf{R}^3 \longrightarrow 0 \tag{40}$$

This sequence is a complex because $\operatorname{curl}\operatorname{grad} f = 0$ and $\operatorname{div}\operatorname{curl}\mathbf{v} = 0$. In fact, it is clear from the definition of the exterior derivative that $d^2 = 0$. This is what is used to define the de Rham cohomology. $\mathbf{R}^3$ isn't such a good example in this context, both because it has trivial cohomology and because it isn't compact. It is a good example of calculating the exterior derivative though and an interesting example because it shows that the different operators in vector calculus are all just examples of one structure. It might

[11] I'm leaving out the half for convenience, thus, in the previous notation $\eta_1 = \frac{1}{2}\eta_{23}$ and so on.

also be interesting to note that the wedge product on $\Omega^1\mathbf{R}^3$ coincides with the usual cross product of vectors.

Our next task is to list the important properties of the exterior derivative and then to observe, after Spivak, that these actually define a unique operator.

We noted above that the exterior derivative sends a function f to df. It is clearly linear and it is easy to see that it acts on exterior products as a graded derivation:

$$d(\eta \wedge \zeta) = d\eta \wedge \zeta + (-)^p \eta \wedge d\zeta \tag{41}$$

where *eta* is a p-form. Finally, as noted above, since differentiation is symmetric and the wedge product of one-forms is skew-symmetric

$$d(d\omega) = 0. \tag{42}$$

If d' is another linear graded derivation with $d'^2 f = 0$ and $d'f = df$ then $d'\omega = d\omega$. This is proven by induction. By linearity we need only consider $\omega = f dx_1 \wedge \ldots \wedge dx_p$. Acting with d',

$$d'\omega = d'f \wedge dx + f \wedge d'(dx_1 \wedge \ldots \wedge dx_p) = df \wedge dx + f \wedge d'(dx_1 \wedge \ldots \wedge dx_p) \tag{43}$$

If we assume that $d\eta = d'\eta$ for ω a $(p-1)$-form then

$$d'dx = d'(dx_1 \wedge dx_2 \wedge \ldots \wedge dx_p) = -d'x_1 \wedge d'(dx_2 \wedge \ldots \wedge dx_p) = 0 \tag{44}$$

and so the second term in the expression for $d'\omega$ vanishes and so the equality holds for p-forms, giving the induction step, proving the theorem and demonstrating that the definition we have used is actually coordinate independent.

The exterior derivative is only one of the differential operators we should be considering on a smooth manifold. Another very common and very important operator is the **Lie derivative**. Unfortunately we need to press on and so the next lecture will start with the de Rham cohomology. This is defined by the complex

$$\ldots \xrightarrow{d} \Omega^p M \xrightarrow{d} \Omega^{p+1} M \xrightarrow{d} \Omega^{p+2} M \longrightarrow \ldots \tag{45}$$