Rough notes for Maths 543

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Lecture 7

The higher homotopy groups

The higher homotopy groups have already been mentioned a few times. They are the groups generated by homotopy classes of maps of the higher-dimensional generalization of a loop. The first thing to do is define the higher-dimensional generalization of a loop. Recall how, when we were studying the fundamental group, we defined the loop as an interval with its end points identified. Well, in the same way, we start here with a unit n-**cube**: $[0,1]^n$.[1] The boundary $\partial[0,1]^n$ is the obvious thing, it is made up of points $(x_1, x_2, \ldots, x_n)$ such that at least one of the x_i is zero or one. An n-**loop** based at x_0 is a map

$$\alpha : [0,1]^n \to M \tag{1}$$

such that $\alpha(x) = x_0$ for all $x \in \partial[0,1]^n$. Homotopy generalizes in an obvious way: based loops α and β are **homotopic** if there exists a continuous map $H : [0,1]^n \times [0,1] \to M$ such that

$$\begin{aligned} H(x_1, \ldots, x_n, 0) &= \alpha(x_1, \ldots, x_n) \\ H(x_1, \ldots, x_n, 1) &= \beta(x_1, \ldots, x_n) \\ H(x_1, \ldots, x_n, t) &= x_0 \qquad \text{if } (x_1, \ldots, x_n) \in \partial[0,1]^n \end{aligned} \tag{2}$$

Finally, we define the multiplication of two n-loops

$$\alpha\beta(x_1, x_2, \ldots, x_n) = \begin{cases} \alpha(2x_1, x_2, \ldots, x_n) & 0 \leq x_1 \leq \frac{1}{2} \\ \beta(2x_1 - 1, x_2, \ldots, x_n) & \frac{1}{2} \leq x_1 \leq 1 \end{cases} \tag{3}$$

and the inverse

$$\alpha^{-1}(x_1, x_2, \ldots, x_n) = \alpha(1 - x_1, x_2, \ldots, x_n). \tag{4}$$

The nth **homotopy group**, $\pi_n(M, x_0)$, is now the quotient of the space of based n-loops by homotopy equivalence. It is a group with group structure given by $[\alpha\beta] = [\alpha][\beta]$.

Properties and examples of higher homotopy groups

The higher homotopy groups are Abelian. There is a nice demonstration of this given in Nash and Sen. You begin with the product of two loops $\alpha\beta$. You can then deform the

[1] This all seems a bit wasteful, maybe we would be better off using simplices to define higher homotopy or n-cubes to define simplicial homology. Notice that Spivak uses singular n-cubes when he is defining integration!

loops so that more of the map maps to the base point. In other words, we thicken the boundaries so that there is some region inside the boundary mapping to the base point. The next step is to switch around the two insides and then thin the boundary again. If you remember that there is a smooth function interpolating between zero and one over a finite interval, it is easy enough to convince yourself that each of these steps is a homotopy.

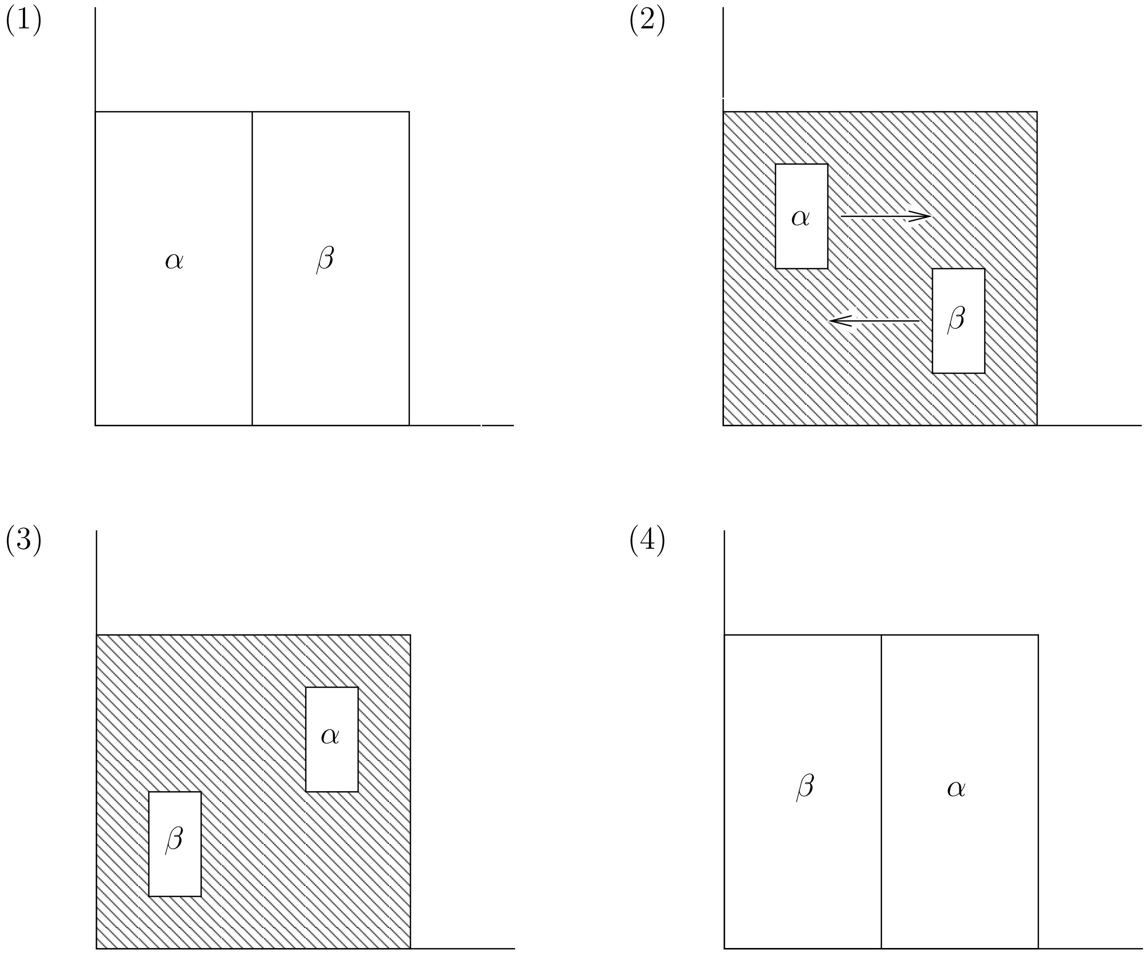

Figure 1: Higher homotopy is Abelian

Calculating higher homotopy is generally hard. Some examples are intuitively clear, a n-sphere for example has $\pi_n(S^n) = \mathbf{Z}$ because an n-loop is homotopically a n-sphere.

You shouldn't be over confident, however. It may not be so clear that $\pi_3(S^2) = \mathbf{Z}$. This charge is called the **Hopf charge** and has a very nice geometrical interpretation. The equivalence classes in $\pi_3(S^2)$ are represented by maps of a two-sphere to a three-sphere. Given a map from a three sphere to a two sphere, any point on the two sphere has a one-dimensional inverse image, this inverse image will be homotopically a circle on the three-sphere. Thus, any pair of points on the two-sphere correspond to a pair of closed curves on the three-sphere. These will have a linking number which counts the number of times they link each other and this number will be the same for any two points. It is the Hopf charge.

Things don't get much better as the numbers get higher, $\pi_6(S^2) = \mathbf{Z}_{12}$ for example. However, the lower homotopy groups for Lie groups are described by the **Bott periodicity theorem**, which states, for example, that, for $n \geq (k+1)/2$

$$\pi_k(\mathrm{U}_n) = \pi_k(\mathrm{SU}_n) = \begin{cases} \text{trivial if } k \text{ is even} \\ \mathbf{Z} \text{ if } k \text{ is odd} \end{cases} \tag{5}$$

In physics, the higher homotopy group is often used to describe topological charges like the winding number associated with the kink in Lecture 2. For example, in the O_3 sigma-model the field is S^2 valued. Thus, an O_3 sigma-model configuration is a map from space to the sphere. For the two-dimensional sigma model the Lagrangian requires that the field approach a constant value at infinity, compactifying space to a two-sphere. This means that the configuration is labeled by an integer in $\pi_2(S^2) = \mathbf{Z}$. Solutions with nonzero charges look like particles known as sigma-model **lumps**. The lumps may play a role in the physics of Quantum Hall conductors. Because of the Hopf charge, the O_3 sigma-model also has non-trivial solutions in three dimensions. These Hopf charged solitons are thought by some people to play a role in QCD, but this is very controversial.

There is another sort of winding numbers associated with solitons. It is often called **winding numbers at infinity**. One example is the Abelian-Higgs vortex. In the Abelian-Higgs theory there is a $\mathbf{C}$-valued scalar field in two dimensional space.[2] The field equations have potential term which vanishes if this field has length one. This means that in a finite energy configuration the length of the field must be one at spatial infinity. Thus, on a very large circle at a great spatial distance the field maps a circle onto something that is topologically a circle.[3] The winding number of this map at large distances is a topological integer, in fact, it counts the number of zeros of the field. The soliton associated with this integer are called **Abelian-Higgs vortex** and is related to the vortices occurring in super-conductors. The solitonic monopoles occurring in gauge theories have a similar topological structure with a winding number in $\pi_2(S^2)$.

Covering spaces

If M and $\tilde{M}$ are connected manifolds then $\tilde{M}$ is a **covering space** for M if there is a continuous surjection p so that if U is an connected open set in M, $p^{-1}(U)$ is a disjoint union of open sets, each of which is homeomorphic to U under p. This is easy to understand by thinking about the covering of a circle by a line

$$\begin{aligned} p : \mathbf{R} &\rightarrow S^1 \\ x &\mapsto x \mod 2\pi \end{aligned} \tag{6}$$

[2]It is a (2+1)-dimensional relativistic field theory, thus, there is a field defined over 2-dimensional space at any given time.

[3]In other words, the field may have some complicated behavior, but for very large r, where $r^2 = x_1^2 + x_2^2$, the field approach length one in some regular way.

so any open interval in the circle is the image of open intervals strung out every 2π along the line.

The useful thing about covering spaces is that they have the same higher homotopy. If $\tilde{M}$ is a cover for M and $\tilde{x}$ maps to x under p then the homomorphism

$$p_* : \pi_n(\tilde{M}, \tilde{x}) \to \pi_n(M, x) \tag{7}$$

induced by p is a isomorphism for $n > 1$. The fundamental groups need not be the same, in the circle example they aren't. If the covering space has trivial fundamental group then it is called a **universal cover**. In this way, $\mathbf{R}$ is the universal cover of S^1 and SU_2 is the universal cover of SO_3.[4]

Relative homotopy

Relative homotopy is a similar idea to relative homology, treated earlier. Relative homotopy groups are generated by relative homology classes of relative loops. A loop is a map of a cube with the faces mapping to a point; for a relative loop, all but one of the faces map to a point and the image of the remaining, exceptional, face is restricted to lie in a specific submanifold of the target manifold. Thus, if N is a closed manifold in M and x_0 is a point in N then a **relative** n-loop is a continuous map $\alpha : [0,1]^n \to M$ such that $\alpha(x) \in N$ is x is one the exceptional face and $\alpha(x) = x_0$ if x is on any of the other faces.[5] The exceptional face is taken to be open, and, so, the faces of the exceptional face are mapped to the base points. This mimics the relative cycle in homology which, considered as a chain in M has a boundary in N.

When we defined a homotopy we required that the base point be fixed throughout the deformation, a **relative homotopy** is the obvious variant of this, it fixes the base point and fixes the fact that the exceptional face maps into N. Thus, taking F to be the exceptional face, two relative n-loops, α and β are homotopic if there exists a continuous map

$$\begin{aligned} H(x_1, \ldots, x_n, 0) &= \alpha(x_1, \ldots, x_n) \\ H(x_1, \ldots, x_n, 1) &= \beta(x_1, \ldots, x_n) \\ H(x_1, \ldots, x_n, t) &= \begin{cases} y \in N & \text{if } (x_1, \ldots, x_n) \in F \\ x_0 & \text{if } (x_1, \ldots, x_n) \in \partial[0,1]^n \setminus F \end{cases} \end{aligned} \tag{8}$$

The **relative homotopy group** $\pi_n(M, N, x_0)$ is the group of homotopy classes of relative n-loops. There are all sort of ways you might worry this would go wrong, how, for example, do you choose which of the faces of a product of loops is the exceptional face. In fact, the relative homotopy define ensures everything works out.

[4]If a manifold is a Lie group, you often refer to the **covering group** to mean the group whose manifold is the universal covering space. These groups, the original group and its cover, have the same Lie algebra because, by the definition of the cover, they are locally identical.

[5]My notation is a bit of a mess, I am using x's and y's both for points in the n-cube and points in the manifold. sorry.

As with homology, the relative homotopy leads to an exact sequence. The sequence is

$$\ldots \longrightarrow \pi_n(N, x_0) \xrightarrow{i_*} \pi_n(M, x_0) \xrightarrow{j_*} \pi_n(M, N, x_0) \xrightarrow{\partial} \pi_{n-1}(N, x_0) \xrightarrow{i_*} \ldots \tag{9}$$

The inclusion map $i_* : \pi_n(N, x_0) \to \pi_n(M, x_0)$ is obvious, since $N \subset M$ a loop in N is a loop in M. The map $j_* : \pi_n(M, x_0) \to \pi_n(M, N, x_0)$ is also an inclusion map: a loop based at $x_0 \in N$ can be regarded as a relative loop since the exceptional face is trivially mapped into N. finally, a relative n-loop has its exceptional face mapped into N. The boundary map $\partial_* : \pi_n(M, N, x_0) \to \pi_{n-1}(N, x_0)$ maps a relative n-loop to the $(n-1)$-loop in N given by the map of the exceptional face. It is easy and instructive to prove the sequence is exact.

In Nash and Sen this exact sequence is used to prove that $\pi_k(S^n)$ is homeomorphic to $\pi_{k+1}(S^{n+1})$. Nash and Sen should be consulted for further details about the calculation of higher homotopy groups. It should be noted that there is no excision theorem for homotopy, this is what makes the calculations more difficult.