Rough notes for Maths 543

Please send corrections and comments to Conor Houghton: `houghton@maths.tcd.ie`

Lecture 6

The long exact sequence of homology groups, continued

The first part of this lecture will be spent proving that

$$\ldots \longrightarrow H_n(N) \xrightarrow{i_*} H_n(M) \xrightarrow{q_*} H_n(M,N) \xrightarrow{\partial} H_{n-1}(N) \xrightarrow{i_*} H_{n-1}(M) \longrightarrow \ldots \tag{1}$$

is a long exact sequence, however, before we begin, it is useful to recall the construction of the boundary operator $\partial : H_n(M,N) \to H_{n-1}(N)$. The construction is summed up by the diagram

$$\begin{array}{ccccc} & & b \in\ C_n(M) & \xrightarrow{q} & c = q(b) \in\ C_n(M,N) \\ & & \downarrow \partial & & \\ a \in\ C_{n-1}(N) & \xrightarrow{i} & \partial b = i(a) \in\ C_{n-1}(M) & & \end{array} \tag{2}$$

c represents an equivalence class in $H_n(M,N)$ and so is a relative cycle. Because q is onto, there is a b in $C_n(M)$ so that $q(b) = c$, however, b may have a boundary ∂b. This boundary must be in $\operatorname{im} i$ because c is a relative cycle. a is also a cycle, but $[a]$ isn't necessarily trivial in $H_{n-1}(N)$, a only corresponds to a boundary in the $C_{n-1}(M)$.

In proving the exactness of the long sequence, the first few inclusions are easy:

$$\operatorname{im} i_* \subseteq \ker q_* \tag{3}$$

follows from the short exact sequence $qi = 0$. The

$$\operatorname{im} q_* \subseteq \ker \partial_* \tag{4}$$

follows from the definition of ∂, because q is surjective, $c \in H_n(M,N)$ can be written as $q(b)$ for some b, but if $[c]$ is in the $\ker q_*$ then b is a cycle and $\partial b = 0$.[1] Finally,

$$\operatorname{im} \partial \subseteq \ker i_* \tag{5}$$

because, from the definition of ∂, $i_*\partial[c] = [\partial b] = 0$

The inclusions going the other way are trickier. First, we will consider

$$\ker q_* \subseteq \operatorname{im} i_*. \tag{6}$$

[1] In other words, c is alway in $\ker q$ because q is surjective, however, b must be a cycle if $[c]$ is in $\ker q_*$. Remember that q_* maps homology classes of cycles whereas q maps chains.

If b represents a homology class in $\ker q_*$ then $q(b)$ is a boundary. Thus, $q(b) = \partial c'$ for some $c' \in C_{n+1}(M,N)$. Since q is surjective, $c' = q(b')$ for some b' in $C_{n+1}(M)$. $q(b-\partial b') = q(b) - q(\partial b') = q(b) - \partial q(b')$ and $\partial q(b') = \partial c' = q(b)$ so $q(b-\partial b') = 0$. Now, we know $\operatorname{im} i = \ker q$ so this means $b - \partial b' = i(a)$ for some $a \in C_{n-1}(N)$. a is a cycle since $i(\partial a) = \partial i(a) = \partial(b - \partial b') = 0$ and i is injective. Finally, $i_*[a] = [b-\partial b'] = [b]$ and the inclusion (**??**) is proved. [2]

Next we consider

$$\ker \partial \subseteq \operatorname{im} q_*. \tag{7}$$

If c is in $\ker \partial$ then the $a \in C_{n-1}(N)$ in the construction of ∂ is a boundary $a = \partial a'$ with $a' \in C_n(N)$. Now, $b - i(a')$ is a cycle because $\partial(b - i(a')) = \partial b - \partial i(a')$ and commuting the ∂ and i we get $\partial(b - i(a')) = \partial b - i(a)$, this is zero by the definition of a.[3] Now $q(b - i(a')) = q(b) - q(i(a')) = q(b) = c$ and so q_* maps $[b - i(a')]$ to $[c]$. This means $[c]$ is in $\operatorname{im} q_*$ proving the inclusion (7).

We finish with

$$\ker i_* \subseteq \operatorname{im} \partial. \tag{8}$$

We consider an element of $\ker i_*$: $a \in C_{n-1}(N)$ such that $i(a) = \partial b$ for some $b \in C_n(M)$. However, this means $q(b)$ is a cycle since $\partial q(b) = q(\partial b) = q(i(a))$ and ∂ takes $[q(b)]$ to $[a]$.[4] In other words, the inclusions maps a to a chain which is a boundary of some b and the image, $q(b)$, of b under quotient must have a as a boundary and hence $[q(b)]$ maps to $[a]$ This proves the inclusion (8) and completes the proof of exactness.

The excision theorem

Before going on to applications of the long exact sequence we state the **excision theorem**. If L is a subset of N such that $\operatorname{clo} L \subset \operatorname{int} M$ [5] then

$$H_r(M,N) = H_r(M \setminus L, M \setminus L) \tag{9}$$

[2]The inclusion is proved using the diagram

$$\begin{array}{ccccc} & & b' \in C_{n+1}(M) & \xrightarrow{q} & c' \in C_{n+1}(M,N) \\ & & \downarrow \partial & & \downarrow \partial \\ a \in C_n(N) & \xrightarrow{i} & b \in C_n(M) & \xrightarrow{q} & C_n(M,N) \end{array}$$

The substance of the proof is this, $[b] \in \ker q_*$ just means that the image of b under q is a boundary. By taking this boundary away, we construct $b - \partial b'$ which is in $\ker q$ and so we can use the short exact sequence to construct a. a maps to $b - \partial b'$ under i, but the equivalence class of $[b - \partial b']$ is just $[b]$.

[3]$\partial b = i(a)$ by definition. In this part of the proof we use the fact c is in $\ker \partial$ to construct two elements in $C_n(M)$ which both map to the same element in $C_{n-1}(M)$.

[4]Don't get confused, there are two ∂s, the boundary operator on chains which sends $q(b)$ to zero and the boundary operator between holonomy groups which sends $[q(b)]$ to $[a]$.

[5]That is, the **closure** of L is in the **interior** of M, I think I am right in thinking the interior of a space is the largest open set it contains, which means it is the union of all the open sets it contains. The closure is the smallest closed set containing a set, this might be more difficult to define since taking the intersection of closed sets doesn't necessarily give a closed set, one guess would be that the closure is the complement of the interior of the complement. Anyway, the idea is that L is sufficiently inside N.

Thus, we can remove a part of the space without affecting the relative homology. This is an important property of homology and part of why is easier to calculate homology than it is to calculate homotopy. The proof of the excision theorem is quite technical and can be found in Hatcher.

An equivalent statement of the excision theorem is

$$H_r(M,N) = H_r(L, L\cap P) \tag{10}$$

where $M=$int $L\cup$int P.

Two examples

We now look at examples where the long exact sequence is used to calculated homology groups. The first involves the **suspension of a manifold**. If M is a manifold the suspension is

$$\Sigma M = (M\times[-1,1])/\sim \tag{11}$$

where $\sim$ identifies all of $M\times\{-1\}$ with a point and $M\times\{1\}$ with another point. A similar construction to this one is the **cone** of a manifold:

$$\Gamma M = (M\times[0,1])/\sim \tag{12}$$

where, here, $\sim$ identifies all the points in $M\times\{1\}$. In fact, the suspension is the union of two cones, the upper cone Γ^+M and the lower cone Γ^-M. The cone of a manifold has an obvious deformation retract to the point:

$$r(t)(m,s)\mapsto(m,s(1-t)) \tag{13}$$

and so $H_p(\Gamma^+M)=H_p(\Gamma^-M)=0$ for $p>0$. This is exploited to prove

$$H_p(M) = H_{p+1}(\Sigma M) \tag{14}$$

To do this we write down two long exact sequence, first $\Gamma^+M\subset\Sigma M$ and

$$\begin{array}{ccccccccc} \ldots & \to & H_{p+1}(\Gamma^+M) & \to & H_{p+1}(\Sigma M) & \to & H_{p+1}(\Sigma M,\Gamma^+M) & \to & H_p(\Gamma^+M) & \to & \ldots \\ & & =0 & & & & & & =0 & & \end{array} \tag{15}$$

is exact, so

$$H_{p+1}(\Sigma M) = H_{p+1}(\Sigma M,\Gamma^+M) \tag{16}$$

There is another exact sequence corresponding to the inclusion $M\subset\Gamma^-M$:

$$\begin{array}{ccccccccc} \ldots & \to & H_{p+1}(\Gamma^-M) & \to & H_{p+1}(\Gamma^-M,M) & \to & H_p(M) & \to & H_p(\Gamma^-M) & \to & \ldots \\ & & =0 & & & & & & =0 & & \end{array} \tag{17}$$

and this means

$$H_{p+1}(\Gamma^-M,M) = H_p(M). \tag{18}$$

Finally, the excision theorem implies

$$H_{p+1}(\Sigma M, \Gamma^+ M)) = H_{p+1}(\Sigma M \setminus \{x\}, \Gamma^+ M \setminus \{x\})) \tag{19}$$

where the point that results from identifying all of $M \times \{1\}$ is x. Now, it is easy to see that $\Sigma M \setminus \{x\}$ has a deformation retract to $\Gamma^+ M$ and $M \setminus \{x\}$ has a deformation retract to M. Thus

$$H_{p+1}(\Sigma M, \Gamma^+ M)) = H_{p+1}(\Gamma^- M, M)). \tag{20}$$

proving (14).

The sphere example is similar, we choose two subsets of S^n: $N = S^n \setminus \{s\}$ and $S = S^n \setminus \{n\}$ where n and s are two distinct points. They could be the north and south pole for example. Both N and S have deformation retracts to the points, N retracts to n and S to s. We can imagine these retracts by imagining a punctured soap bubble. $N \cap S$ has a deformation retract to S^{n-1}. Now $S^n = N \cup S$ so

$$\begin{array}{ccccccccc} \ldots & \rightarrow & H_{p+1}(N) & \rightarrow & H_{p+1}(S^n) & \rightarrow & H_{p+1}(S^n, N) & \rightarrow & H_p(N) & \rightarrow & \ldots \\ & & = 0 & & & & & & = 0 & & \end{array} \tag{21}$$

so $H_{p+1}(S^n) = H_{p+1}(S^n, N)$ and

$$\begin{array}{ccccccccc} \ldots & \rightarrow & H_{p+1}(S) & \rightarrow & H_{p+1}(S, S \cup N) & \rightarrow & H_p(S \cup N) & \rightarrow & H_p(S) & \rightarrow & \ldots \\ & & = 0 & & & & & & = 0 & & \end{array} \tag{22}$$

so $H_{p+1}(S, S \cup N) = H_p(S \cup N)$. Finally, the excision theorem means that $H_{p+1}(S^n, N) = H_{p+1}(S^n \setminus n, N \setminus n$ and by deformation retract this means $H_{p+1}(S^n, N) = H_{p+1}(S, N \cup S)$. Putting all this together proves

$$H_{p+1}(S^{n+1}) = H_p(S^n) \tag{23}$$

where once again the theorem doesn't apply to $p = 0$ since the zeroth homology group of the point isn't trivial.[6]

The Mayer-Vietoris sequence.

The **Mayer-Vietoris sequence** is another exact sequence of homology groups. It is similar to the long exact sequence discussed above and one or other is more useful depending on the precise problem being studied. The Mayer-Vietoris sequence is defined using the short exact sequence of chain groups

$$0 \to C_r(N \cap L) \xrightarrow{\alpha} C_r(N) \oplus C_r(L) \xrightarrow{\beta} C_r(N + L) \to 0 \tag{24}$$

[6] In fact, it is often convenient to use **reduced homology groups** where ∂_0 is replaced by $\epsilon : C_0(M) \to \mathbf{Z}$ with $\epsilon : (\Sigma_i n_i \sigma_i) \mapsto \Sigma_i n_i$. The advantage of this is that the reduced homology group of a point is trivial for all dimensions, including zero.

where N and L are two subsets of M such that $M = \text{int}\, N \cup \text{int}\, L$ and $C_r(N+L)$ is the subgroup of $C_r(M)$ consisting of chains which are a sum of a chain in N and a chain in L. This subgroup forms a chain complex because $\partial : C_r(N+L) \to C_{r-1}(N+L)$.[7] The correspond homology groups are actually $H_r(M)$ because, as can be proved, any cycle in $C_r(M)$ is homologous to a cycle of the form $a+b$ where $a \in C_r(N)$ and $b \in C_r(L)$. The two maps α and β above are

$$\begin{array}{rcl} \alpha : C_r(N \cap M) & \to & C_r(N) \oplus C_r(L) \\ c & \mapsto & (c, -c) \end{array} \tag{25}$$

and

$$\begin{array}{rcl} \alpha : C_r(N) \oplus C_r(L) & \to & C_r(N+L) \\ (a,b) & \mapsto & a+b. \end{array} \tag{26}$$

Now elements of the homology group $H_r(M)$ are represented by cycles c in $C_r(M)$. A cycle in $C_r(M)$ can alway be written $c = a + b$ with a a chain in N and b a chain in L. However, while $a+b$ is a cycle, a and b are not, they may have boundaries so long as $\partial a = -\partial b$. ∂a is a cycle because it is a boundary in $C_{r-1}(N)$, however, it is not in general a boundary in $C_{r-1}(N \cap L)$. Thus, it defines a homology class in $H_{r-1}(N \cap L)$. This gives us the boundary map:

$$\begin{array}{rcl} \partial : H_r(M) & \longrightarrow & H_{r-1}(N \cap L) \\ c & \mapsto & \partial a \end{array} \tag{27}$$

We now have a sequence

$$\ldots \to H_r(N \cap L) \xrightarrow{\alpha_*} H_r(N) \oplus H_r(L) \xrightarrow{\beta_*} H_r(M) \xrightarrow{\partial} H_{r-1}(N \cap L) \to \ldots \tag{28}$$

It is not too hard to prove that this is an exact sequence.

As as example of applying the Mayer-Vietoris sequence, consider the formula for the homology of the n-sphere derived above. Splitting the n-sphere up into the same subsets as before and substituting into the Mayer-Vietoris sequence gives the same formula without any need to use excision.

Singular Homology

The idea behind **singular homology** is to define homology in a way that is manifestly homeomorphism invariant and in a way that applies to spaces that have no triangulation.

[7] Now I am loosing track, did I define a **chain complex**? A chain complex is the sequence

$$\ldots \to C_{r+1} \xrightarrow{\partial} C_r \xrightarrow{\partial} C_{r-1} \to \ldots$$

that is used to define homology.

A singular n-simplex is a continuous map[8] of an n-simplex into the space

$$\lambda_r : \sigma_r \to M \tag{29}$$

and the chain group is the free Abelian group generated by the singular n-simplices. Obviously this group can be very large since there can be uncountably many singular n-simplices in a manifold. However, the boundary operator can be defined as before, the boundary of a singular simplex λ is the image under λ of the boundary of the n-simplex σ being mapped.[9] This boundary map still squares to zero and so the singular chain group defined homology groups. These homology groups are the singular homology. It can be shown that the singular homology group of a triangular space is the same as the simplicial homology.

[8]but not necessarily invertible, hence the 'singular' in the name.

[9]In other words, it is the sum of the images of the faces of σ with the usual plus and minus ones.