Rough notes for Maths 543

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Lecture 5

The direct calculation of homology groups

It is obvious from the definition of the homology groups how the groups may be calculated directly. All you need to do is take a general element of the chain group and work out it cycles and the boundaries.

The first example is the circle S^1 with triangulating simplicial complex

$$\{\langle a_1a_2\rangle, \langle a_2a_3\rangle, \langle a_3a_1\rangle, a_1, a_2, a_3\}. \tag{1}$$

Since a 0-chain has no boundary, the three 0-cycles $\{a_1, a_2, a_3\}$ form a basis of the group of 0-cycles. However any two of these points are homologous and so $H_0(S^1) = \mathbf{Z}$.[1] A general 1-chain is of the form

$$c = x\langle a_1a_2\rangle + y\langle a_2a_3\rangle + z\langle a_3a_1\rangle \tag{2}$$

and, since

$$\partial c = (x-z)a_1 + (y-x)a_2 + (z-y)a_3 \tag{3}$$

c is a 1-cycle provided $x = y = z$. There are no 2-chains and so the first holonomy group is the group generated by this 1-cycle. Thus, $H_1(S^1) = \mathbf{Z}$ as well.

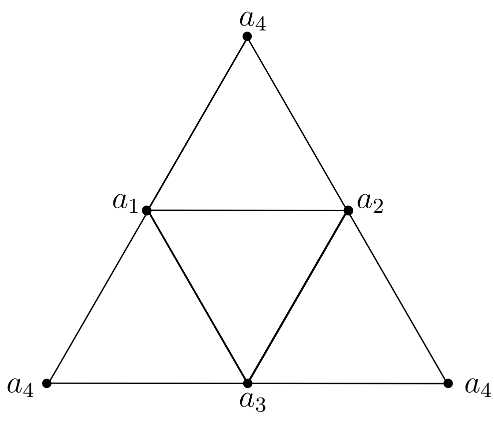

Figure 1: A triangulation of the two-sphere.

[1] In fact, this is a particular case of a general feature: if M is a path-wise connect space then $H_0(M) = \mathbf{Z}$. This is because any two points are path connected and that path provides a homology between the points.

As the next example consider the 2-sphere. The 2-sphere is triangulated by thr faces of a 3-simplex, this is illustrated in Fig. 1. Since the 2-sphere is simply connected $H_0(S^2) = \mathbf{Z}$. Next, consider a general 1-chain

$$c = x_1\langle a_1a_2\rangle + x_2\langle a_2a_3\rangle + x_3\langle a_3a_1\rangle + x_4\langle a_1a_4\rangle + x_5\langle a_2a_4\rangle + x_6\langle a_3a_4\rangle. \tag{4}$$

The boundary is

$$\partial c = (x_1 + x_4 - x_3)a_1 + (-x_1 + x_2 + x_5)a_2 + (-x_2 + x_3 + x_6)a_3 - (x_4 + x_5 + x_6)a_4. \tag{5}$$

For c to be a 1-cycle this must be zero. However, every variable appear in two equations, once with one sign and once with the other and so the four equations add to zero: there are only three independent constraints. Thus, the group is 1-cycles is four dimensional. This means that $\{\partial\langle a_1a_2a_3\rangle, \partial\langle a_1a_4a_2\rangle, \partial\langle a_1a_3a_4\rangle, \partial\langle a_2a_3a_4\rangle\}$ is a basis for the cycles. Thus each member of the basis set is a boundary and $H_1(S^2)$ is trivial. Finally, a general 2-chain is

$$c = x_1\langle a_1a_2a_3\rangle + x_2\langle a_1a_4a_2\rangle + x_3\langle a_1a_3a_4\rangle + x_4\langle a_3a_2a_4\rangle. \tag{6}$$

This has boundary

$$\begin{aligned} \partial c &= (x_1 - x_2)\langle a_1a_2\rangle + (x_1 - x_4)\langle a_2a_3\rangle + (x_1 - x_3)\langle a_3a_1\rangle \\ &+ (x_2 - x_3)\langle a_1a_4\rangle + (-x_2 + x_4)\langle a_2a_4\rangle + (x_3 - x_4)\langle a_3a_4\rangle \end{aligned} \tag{7}$$

and so, for c to be a cycle, $x_1 = x_2 = x_3 = x_4$ and therefore $H_2(S^2) = \mathbf{Z}$.

This example was done is an needlessly long-winded way, however, it should be obvious that calculating the homology groups of some of the spaces we triangulated would be quite a lengthy process. Luckily, it is not necessary to use the triangulation in calculating the homology of a space. In fact, it is sufficient to find a **Δ-complex** which is homeomorphic to the space.[2]

A Δ-complex is like a simplicial complex without such a strong intersection condition. It is a set of simplices such that any face of any simplex in the Δ-complex is in the Δ-complex and such that any two simplices intersect along faces. In other words, if two simplices in the complex intersect, they must do so on some of the faces, but they are allowed to do so on more than one face.[3] Anyway, it seems that any Δ-complex can be subdivided to give a homeomorphic simplicial complex and so homology can be defined using Δ-complexes rather than simplicial complexes and the same groups will be the result.[4]

[2]The name Δ-complex reflects the fact that simplices are often called Δ, as opposed to σ, this is an obvious choice of notation since Δ looks like a triangle, I'm sorry I didn't use it.

[3]There is also some understanding about being careful about orientation, the complex is regarded as a set of oriented simplices, all the faces of an oriented simplex have an orientation, a face is the simplex whose vertices are a subset of the vertices of the simplex, the orientation of the face is given by having the vertices in the subset in the same order as in the simplex. In the Δ-complex, it is required that a face in the intersection of two oriented simplices is an oriented face of each of them. This just means that, two simplices can only be joined along faces that are oriented the same way.

[4]You might wonder then why we used simplicial complexes at all, well the nice thing about simplicial complexes is that you can name a simplex in a simplicial complex by giving its vertices, that isn't the case for a Δ-complex. This means that it is nicer thinking about holonomy in terms of simplicial complexes unless you are thinking about doing actual calculations. The question is, can you work out homotopy groups using Δ-complexes, I haven't thought about that yet.

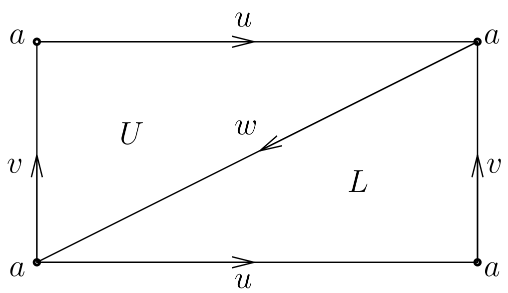

Figure 2: The Δ-complex for a torus, since the simplices are no longer uniquely determined by their vertices it is neccessary to name them, the names should be read off this diagram, the names for the two 2-simplices, U and L are chosen to demonstrate my debt to Hatcher in this part of the course.

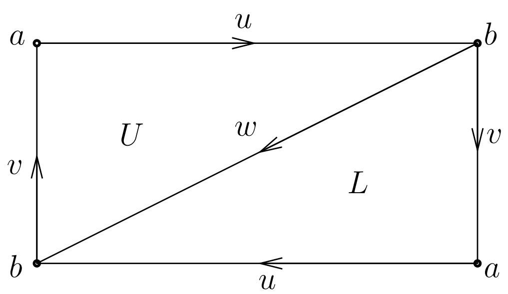

Figure 3: The Δ-complex for the real projective plane.

The Δ-complex for the torus is given in Fig. 2. The two 2-simplices have the same boundary, $\partial U = \partial L = u+v+w$ and so $U-L$ is a basis for the 2-cycles and $H_2(T^2) = \mathbf{Z}$. The three 1-simplices u, v and w are all cycles and $u+v+w$ is a boundary, so $H_1(T^2) = \mathbf{Z}\oplus\mathbf{Z}$. $H_0(T^2) = \mathbf{Z}$ in the usual way.

The Δ-complex for the real projective plane is given in Fig. 3. In this case $\partial U = u+v+w$ as before, but $\partial L = w-u-v$ and so there are no 2-cycles. This means that $H_2(\mathbf{RP}^2) = 0$. The 1-cycles are w and $u+v$ whereas the 1-boundaries are $u+v+w$ and $w-u-v$. Thus, w is homologous to $u+v$ and $2w = (u+v+w)+(w-u-v)$ is homologous to zero: $H_1(\mathbf{RP}^2) = \mathbf{Z}_2$. The Δ-complex for the Klein bottle, K, is given in Fig. 4. It is easy to see that $H_1(K) = \mathbf{Z}\oplus\mathbf{Z}_2$.

Another nice thing we can do easily with Δ-complexes is work out $H_n(S^n)$. We can make a Δ-complex for an n-sphere by identifying all the faces of two n-simplices. There is only one n-cycle given by the difference of these two simplices. Thus, $H_n(S^n) = \mathbf{Z}$.

An interesting example is the genus g surface, Γ_g. Unfortunately it is very difficult to

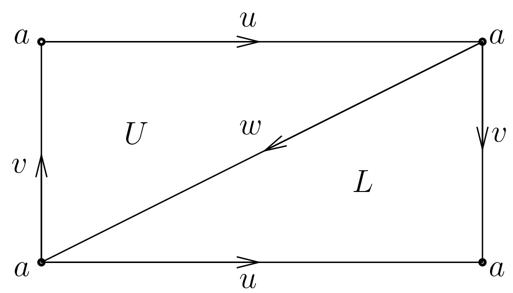

Figure 4: The Δ-complex for the Klein bottle.

draw using `xfig` so I will refer you to Farkas and Kra and also to Hatcher. The important point is that $H_1(\Gamma_g) = \mathbf{Z}^{2g}$ and there is a **canonical basis** for the holonomy cycles $\{a_i, b_i\}$ such that the only non-trivial intersection is $a_i \cap b_i$. We may return to Riemann surfaces later in the course.

Torsion, homology with real coefficients

A homology groups is a quotient of two free Abelian groups. This means that homology groups have the general structure

$$H_r(M) = \mathbf{Z} \oplus \ldots \oplus \mathbf{Z} \oplus \mathbf{Z}_{k_1} \oplus \ldots \oplus \mathbf{Z}_{k_p} \tag{8}$$

where the k_is are all integers. The picture we have of homology is that the number of **Z**s counts the number of holes in the space and the cyclic groups describe how the space is twisted around. In fact, the part of the group made up of cyclic groups is called the **torsion group**. The number of bfZ is the rth **Betti number**.[5]

It is possible to define homology with different coefficients, we can write an element c of the chain group C_r with coefficients in some other Abelian group G,

$$c = \sum g_i \sigma_i \tag{9}$$

where the σ_is are r-simplices and the g_is are in G. The boundary operator acts on simplices and this action is extended to the whole chain group by linearity, this means it act in the same way on chain groups with coefficents in some other group and so r-cycles and r-boundaries can be defined. This leads to a homology group $H_r(M;G)$ which is generated over G by homology classes of cycles. Since the homology depends on the homology classes of cycles, using a different group doesn't give us extra information, in fact, if G doesn't have

[5]In Hatcher it is pointed out that the Betti numbers and the torsion coefficiants k_i are older than the homology groups, they were defined using simplicies and boundary operators without the homology groups themselves begin considered.

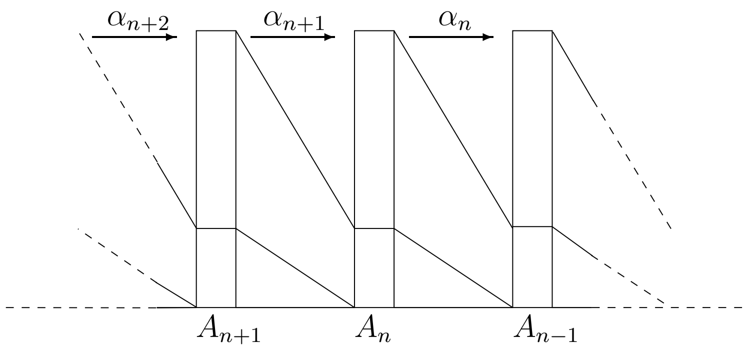

Figure 5: Everything in a kernel gets mapped to zero.

normal subgroup there can be no torsion. Put another way, $\mathbf{R}/\mathbf{R}$ is the identity, whereas $\mathbf{Z}/\mathbf{Z}$ may be some $\mathbf{Z}_k$ depending on how one $\mathbf{Z}$ is contained in the other. If $H_r(M;\mathbf{Z})$ in (8) has b_r factors of $\mathbf{Z}$ then $H_r(M;\mathbf{R}) = \mathbf{R} \oplus \ldots \oplus \mathbf{R}$ with b_r factors of $\mathbf{R}$.

Although it is not sensitive to torsion, homology over $\mathbf{R}$ rather than $\mathbf{Z}$ is useful because it is homology over $\mathbf{R}$ that is dual to cohomology.

A long exact sequence of homology groups

We will not see how homology groups of various dimensions and the homology of a space and its subspaces can be related. To do this, we must first look at **exact sequences** and we must also define **relative homology**.[6]

Introducing exact sequences

A sequence of group homeomorphisms

$$\ldots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \ldots \tag{10}$$

is exact if

$$\ker \alpha_n = \operatorname{im} \alpha_{n+1} \tag{11}$$

This means that $\alpha_n \alpha_{n+1} = 0$ but that the corresponding homology group is trivial. For those of us who enjoy diagrams, exactness is illustrated in Fig. 5.

Next, we consider some special cases,

$$0 \longrightarrow A \xrightarrow{\alpha} B \tag{12}$$

[6]It might initially seem that these are two different thing, they aren't though, the relative homology groups are related to group quotients and there are short exact sequences of groups, their subgroups and their quotients.

is exact if $\ker\alpha = 0$,[7] in other words, if α is injective.[8]

$$A \xrightarrow{\alpha} B \longrightarrow 0 \tag{13}$$

on the other hand, is exact if $\operatorname{im}\alpha = B$, in other words, if α is surjective.[9] Putting these together

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0 \tag{14}$$

is exact if α is an isomorphism.

We can now define a **short exact sequence**. Consider

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \tag{15}$$

If this sequence is exact, α is injective so $\operatorname{im}\alpha$ is homeomorphic to A. β is surjective, so C is homeomorphic to $B/\operatorname{im}\alpha$. This is a short exact sequence. In particular, if A is a normal subgroup of B and i is the inclusion of A in B then

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{q} B/A \longrightarrow 0 \tag{16}$$

is a short exact sequence with i the inclusion map and q the quotient map which identifies all the elements in an equivalence class.

Relative homology

The short exact sequence will be used to define an exact sequence of homology groups. Before doing this, we need to define the **relative homology**. If $N \subset M$ then $C_n(N)$ is a subgroup[10] of $C_n(M)$. TThe boundary operator

$$\partial : C_n(M) \to C_{n-1}(M) \tag{17}$$

acts on the chain groups of N:

$$\partial : C_n(N) \to C_{n-1}(N) \tag{18}$$

because any chain on N has its boundary on N. This means the boundary operator is well defined on the quotient

$$\partial : C_n(M,N) \to C_{n-1}(M,N) \tag{19}$$

where $C_r(M,N) = C_r(M)/C_r(N)$. The cycles in $C_n(M,N)$ include chains with boundaries in N, the boundaries in $C_n(M,N)$ are not necessarily boundaries in $C_n(M)$, however, it is still true that $\partial^2 = 0$ and so the relative homology group can be defined

$$H_n(M,N) = \ker\partial/\operatorname{im}\partial. \tag{20}$$

[7] 0 is include, so 0 maps to 0, the image of the first map being trivial means the kernel of the second one must be trivial as well.

[8] 1 to 1 in english.

[9] Onto in english.

[10] A normal subgroup, but everything is Abelian so this isn't such a delicate issue.

The long exact sequence of homology groups

There is a short exact sequence of chain groups given by

$$0 \longrightarrow C_n(N) \xrightarrow{i} C_n(M) \xrightarrow{q} C_n(M,N) \longrightarrow 0 \tag{21}$$

Furthermore the squares

$$\begin{array}{ccc} C_n(N) & \xrightarrow{i} & C_n(M) \\ \downarrow \partial & & \downarrow \partial \\ C_{n-1}(N) & \xrightarrow{i} & C_{n-1}(M) \end{array} \tag{22}$$

and

$$\begin{array}{ccc} C_n(M) & \xrightarrow{q} & C_n(M,N) \\ \downarrow \partial & & \downarrow \partial \\ C_{n-1}(M) & \xrightarrow{q} & C_{n-1}(M,N) \end{array} \tag{23}$$

commute.[11] This means that i and q induce maps on the homology groups, called i_* and q_*.

Before defining the long exact sequence we need to define an operator which maps a relative cycle to its boundary. This map is called ∂ and

$$\partial : H_n(M,N) \to H_{n-1}(M) \tag{24}$$

To see how to construct the image, $\partial[c]$, of the homology class of some relative cycle c, we stack two short exact sequences on top of one another:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n(N) & \xrightarrow{i} & C_n(M) & \xrightarrow{q} & C_n(M,N) & \longrightarrow & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ 0 & \longrightarrow & C_{n-1}(N) & \xrightarrow{i} & C_{n-1}(M) & \xrightarrow{q} & C_{n-1}(M,N) & \longrightarrow & 0 \end{array} \tag{25}$$

and extract the sub-diagram

$$\begin{array}{cccc} & & C_n(M) & \xrightarrow{q} & C_n(M,N) \\ & & \downarrow \partial & & \\ C_{n-1}(N) & \xrightarrow{i} & C_{n-1}(M) & & \end{array} \tag{26}$$

where, as usual, q is surjective and i is injective. Now, consider a cycle, c, in $C_n(M,N)$. Because q is surjective, there is a b in $C_n(M)$ such that $q(b) = c$. This b can be mapped to element ∂b in $H_{n-1}(M)$. Now, the relevant square commutes and $q(\partial b) = \partial q(b) = \partial c = 0$ since c is a cycle. This means that ∂b is in the $\ker q$ and, since by exactness $\ker q = \operatorname{im} i$, $\partial b \in \operatorname{im} i$. Thus, there is an $a \in C_{n-1}(N)$ such that $i(a) = \partial b$. This a is a cycle: by commutativity of the relevant square $i(\partial a) = \partial i(a) = \partial\partial b = 0$ and, because i is injective,

[11] That is, $\partial i = i\partial$ and $\partial q = q\partial$ acting on the appropriate groups, to say a square commutes means that the two routes from the corner with all the arrows leaving to the kitty corver with all the arrows arriving give the same answer.

this implies $\partial a = 0$. Thus, we have associated a cycle in $C_{n-1}(N)$ with a cycle in $C_n(M,N)$ and, so, we define the map

$$\partial : [c] \mapsto [a] \tag{27}$$

Obviously we should check this map is well defined, this is easy, interesting and fun, it is also done in Hatcher. By tracing the construction of a as it moves back along the diagram we see the meaning of the map: since c is a cycle in the relative chain group, it may have a boundary, but this boundary must lie in N. However, this boundary must by a cycle. ∂ maps the homology class of the relative cycle c to the homology class of its boundary.

These maps are used to define the long exact sequence of homology groups

$$\ldots \longrightarrow H_n(N) \xrightarrow{i_*} H_n(M) \xrightarrow{q_*} H_n(M,N) \xrightarrow{\partial} H_{n-1}(N) \xrightarrow{i_*} H_{n-1}(M) \longrightarrow \ldots \tag{28}$$

The exactness of this sequence will be established next week.