Rough notes for Maths 543

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Lecture 4

It has previously been remarked that homotopy isn't as useful as you'd at first think. The fundamental group generalizes to higher homotopy groups, the groups generated by the homotopy classes of spheres to the manifold. These higher homotopy groups are sensitive to topological features that do not affect the fundemental group. Consider a 2-sphere, it has a trivial fundamental group, the loops cannot wind around the interesting topology, but it has $\pi_2(S^2) = \mathbf{Z}$ because a map of a 2-sphere does wind around the topology.[1] However, these higher homotopy groups are hard to calculate and are not an easy way to calculated topological properties of a manifold.[2] Homology groups are like homotopy groups in that they are non-trivial homeomorphism invariant groups generated by subspaces of a manifold. They are easier to calculate than homotopy groups.

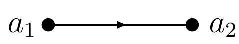
One of the easiest ways of defining homology is in terms of the triangulations and simplicial complexes introduced previously.[3]

Oriented simplices and boundary operators

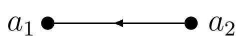
When we introduced simplices we weren't very careful about orientation. This was silly really, when we actually used the triangulation, the orientation was important: a group generator was associated with one orientation and its inverse with the other. Simplicial homology also relies on groups generated by simplices and so it makes sense to think of simplices as group objects and write

$$\langle a_2 a_1 \rangle = -\langle a_1 a_2 \rangle. \tag{1}$$

In other words, the oriented simplex



is minus the simplex



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More generally we define

$$\langle a_1 a_2 \dots a_n \rangle = (-)^s \langle a_i a_j \dots a_k \rangle \tag{2}$$

[1] It is interesting that $\pi_2(S^3) = \mathbf{Z}$ as well, we will discuss this later.

[2] The higher homotopy groups do have an intuitive appeal and physicist often talk about winding numbers for higher homotopy. Calculating higher homotopy groups is also an interesting problem in its own right and will probably be discussed later in this course.

[3] But not the only way, there are many different definitions, each useful in different situations

where

$$s = \text{sign}\begin{pmatrix} 1 & 2 & \dots & n \\ i & j & \dots & k \end{pmatrix} \tag{3}$$

Thus

$$\langle a_1a_2a_3\rangle = \langle a_2a_3a_1\rangle = \langle a_3a_1a_2\rangle \tag{4}$$

and

$$\langle a_2a_1a_3\rangle = \langle a_3a_2a_1\rangle = \langle a_1a_3a_2\rangle \tag{5}$$

but

$$\langle a_1a_2a_3\rangle = -\langle a_2a_1a_3\rangle. \tag{6}$$

The idea behind orienting the simplex is to define the **chain group**. Recall that a simplicial complex is a collection of simplices satisfying two conditions, first, every face of any simplex in the complex is itself in the complex and, second, if two simplices intersect, they intersect along a single shared face. The rth chain group C_r of a simplicial complex is the Abelian group[4] generated by the r-simplices. In other words, elements of the chain group are of the form

$$c = \sum_i c_i\sigma_i \tag{7}$$

where $\{\sigma_i\}$ are the r-simplices in the simplicial complex and $c_i \in \mathbf{Z}$. The group operation is

$$\sum_i a_i\sigma_i + \sum_i b_i\sigma_i = \sum_i (a_i + b_i)\sigma_i. \tag{8}$$

Thus, as a group

$$C_r = \mathbf{Z} \oplus \mathbf{Z} \oplus \dots \oplus \mathbf{Z} \tag{9}$$

where the number of **Z**s is given by the number of unoriented r-simplices in the complex.

We see again how useful triangulation is, we started with a space, a very tricky thing, and we have reduced it to a set of algebraic objects, the chain groups. However, the chain groups are not themselves useful, they depend on the triangulation. The idea is to find elements in the chain groups which play an analogous role to the loops in homotopy theory. The loops in homotopy theory could be wrapped around certain topological features in the space and so they were a good probe for certain aspects of the topology. The analogous object in homology is a subspace with no boundary; our next task is to define the boundary operator.[5]

The **boundary map** maps an oriented simplex to its oriented faces.

$$\partial : \langle a_0a_1\dots a_n\rangle \to \sum_{i=0}^{n}(-)^i\langle a_0a_1\dots\hat{a}_i\dots a_n\rangle \tag{10}$$

[4]Note that this is different to what we did when we were calculating homotopy, the chain groups are Abelian, the edge groups where not.

[5]Of course, in the case of simplicial homology, we don't deal with r-dimensional subspaces but with collections of r-simplices. In other words, homology is about certain equivalence classes of subspaces without boundaries but to define this precisely, we don't deal with manifolds, we deal with simplicial complexes and we don't deal with subspaces, we deal with collections of simplices.

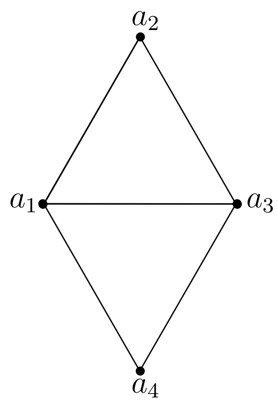


Figure 1: The 1-simplex $\langle a_1 a_3 \rangle$ is not in the boundary of the 1-chain $\langle a_1 a_2 a_3 \rangle + \langle a_1 a_3 a_4 \rangle$.

where $\langle a_0 a_1 \ldots \widehat{a_i} \ldots a_n \rangle$ means the simplex with a_i omitted, so, for example, $\langle a_1 \widehat{a_2} a_3 \rangle = \langle a_1 a_3 \rangle$.[6] Thus,

$$\partial \langle a_1 a_2 a_3 \rangle = \langle a_1 a_2 \rangle - \langle a_1 a_3 \rangle + \langle a_2 a_3 \rangle \tag{11}$$

or

$$\partial \langle a_1 a_2 \rangle = \langle a_1 \rangle - \langle a_2 \rangle. \tag{12}$$

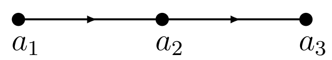
By linearity this definition extends to a map on elements of the chain group,

$$\partial \sum_i c_i \sigma_i = \sum_i c_i \partial \sigma_i. \tag{13}$$

It maps from r-chains to $(r-1)$-chains. [7] Writing ∂_r for ∂ acting on r-chains, we have

$$\partial_r : C_r \to C_{r-1} \tag{14}$$

A nice thing about this definition of the boundary operator is that it is an algebraic operation on the simplices, but it nonetheless agrees with what we might think the boundary operator should do. Take, for example,



we have

$$\partial(\langle a_1 a_2 \rangle + \langle a_2 a_3 \rangle) = a_1 - a_2 + a_2 - a_3 = a_1 - a_3 \tag{15}$$

and so the boundary operator isn't fooled by the extra point, a_2. A similar example is given in Fig. 1. Here

$$\partial(\langle a_1 a_2 a_3 \rangle + \langle a_1 a_3 a_4 \rangle = \langle a_1 a_2 \rangle + \langle a_2 a_3 \rangle + \langle a_3 a_4 \rangle + \langle a_4 a_1 \rangle \tag{16}$$

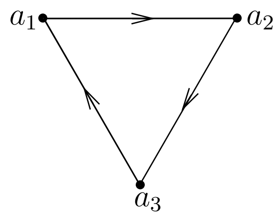


Figure 2: $\langle a_1a_2\rangle + \langle a_2a_3\rangle + \langle a_3a_1\rangle$ is a 2-chain with no boundary.

and so the boundary operator has mapped the 2-chain to its boundary.

The useful thing about this definition of the boundary operator is that something that hasn't a boundary in the normal sense, hasn't a boundary in this sense either. Consider Fig. 2,

$$\partial(\langle a_1a_2\rangle + \langle a_2a_3\rangle + \langle a_3a_1\rangle) = 0 \tag{17}$$

It should also be noted that

$$\langle a_1a_2\rangle + \langle a_2a_3\rangle + \langle a_3a_1\rangle = \partial\langle a_1a_2a_3\rangle. \tag{18}$$

So, $\langle a_1a_2\rangle + \langle a_2a_3\rangle + \langle a_3a_1\rangle$ is a 2-chain with no boundary and, is, furthermore, itself the boundary of a 2-chain.

This is an example of a more general principal: the boundary of a boundary has no boundary. With a bit of thought, it is obvious that this is true of a subspace, if something covers something else it can't itself have a boundary. This can be seen to be true of the boundary operator here by explicit calculation. Since the boundary operator is linear, we need only consider a simplex and

$$\begin{aligned}
\partial^2\langle a_0a_1\dots a_n\rangle &= \partial\sum_i(-)^i\langle a_0a_1\dots\widehat{a_i}\dots a_n\rangle \\
&= \sum_{j\geq i}(-)^{j+1}\sum_i(-)^i\langle a_0a_1\dots\widehat{a_i}\dots\widehat{a_{j+1}}\dots a_n\rangle \\
&+ \sum_{j<i}(-)^j\sum_i(-)^i\langle a_0a_1\dots\widehat{a_j}\dots\widehat{a_i}\dots a_n\rangle \\
&= 0.
\end{aligned} \tag{19}$$

where the double sums have been written so that the same two a_i's are being omitted in each summand. In other words, any $(n-2)$-simplex, $\langle a_0a_1\dots\widehat{a_r}\dots\widehat{a_s}\dots a_n\rangle$, occurs twice in the sum on the right hand. It occurs once because a_s has been removed by the first boundary operator and a_r by the second. In this case, it has a prefactor of $(-)^{r+s}$. It also

[6] On a notational point, the notation used for the simplices is not universal, different authors use different notation. The notation with the hat though, is very common.

[7] By an r-**chain** I mean an element of C_r, the expression makes pictorial sense.

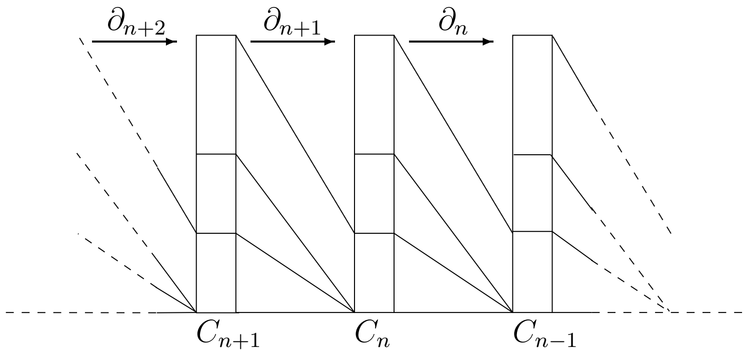


Figure 3: The idea here is that the tower blocks represent chain groups and the horizontal lines subdivide these; everything up to the bottom subdivision is a boundary, everything up to the next subdivision is cycle. I find this diagram helpful, ignore it if you do not.

occurs because a_r is removed by the first boundary operator and a_s by the second. In this case, the a_s isn't the $s+1$th point when it is removed, because the a_r element is already gone, it is the sth point and the prefactor is $(-)^{r+s+1}$. This leads to the cancellation and hence to $\partial^2\sigma = 0$.[8]

This means that the subgroup of the chain group of chains without boundaries contains as a subgroup the chains which are themselves boundaries.[9] It is easier to say this if we first define an r-cycle as a r-chain without a boundary, that is, $c \in C_r$ and $\partial c = 0$. We also define an r-**boundary** as an r-cycle which is a boundary, that is, it is of the form ∂c where $c \in C_{r+1}$. Using this new terminology, the fact $\partial^2\sigma = 0$ means that every r-boundary is an r-cycle. This is summed up with an idiosyncratic diagram in Fig. 3.

The definition of homology

The reason we are interested in r-boundaries and r-cycles is that they allow us to define a group: the **homology** group, which encodes useful information about the topology of the triangulated space. The reason r-cycles are important is that they are like the loops used in homotopy theory, they have no boundary. This means that they can can be wrapped around some topological feature of the space. For this to make sense we must be considering some equivalence class of cycles. We want to be able to say that such and such a cycle is equivalent to the null cycle, whereas this other cycle is not because there is an obstruction. In the fundemental group, the equivalence was homotopy, here the same role is played by homology, we say that two r-cycles are **homologous** if they are the boundary of some $r+1$-cycle. By imagining how the homotopy acting on a loop sweeps out a surface, it is

[8]This is sloppy notation, it isn't the same boundary operator in each case, because the two ∂s have different ranges and domains. What I mean is $\partial_{n+1}\partial_n\sigma = 0$.

[9]I slipped in the subgroup bit, I only proved that it was a subspace, the group properties are easy enough to verify.

easy to see that the idea behind homology is similar to the one behind homotopy.[10]

Anyway, the r-homotopy group of some simplicial complex K is defined as the quotient of the group of r-cycles by the group of r-boundaries:

$$H_r(K) = \ker \partial_r / \mathrm{im}\, \partial_{r+1} \tag{20}$$

It is invariant under homeomorphism, this means it doesn't depend on the triangulation and topologically equivalent spaces have the same homology groups.

Of course, for the definition to work for $H_n(K)$ where n is the dimension of $\mathbf{K}$, we need to let $C_{n+1} = 0$ and define $\partial_{n+1} 0 = 0$. This means $H_n(K) = \ker \partial_n$. Similarly, $H_0(K) = C_0 / \mathrm{im}\, \partial_1$. In fact, C_0 is the chain group generated by points and the quotient makes two points equivalent if they are connect by a 1-chain. In other words, for a path connected manifold, $H_0(M) = \mathbf{Z}$.

[10] But not identical of course.