Rough notes for Maths 543

Please send corrections and comments to Conor Houghton: houghton@maths.tcd.ie

Lecture 3

The sine-Gordon Kink

Winding numbers are very important in the theory of solitons. The sine-Gordon kink is an example of a soliton.[1] The sine-Gordon model is a bosonic field theory in 1+1-dimensions with a periodic Lagrangian[2]

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - (1-\cos\phi). \tag{1}$$

A field is in the **vacuum state** if it is constant in x and t and minimizes the potential. In the sine-Gordon model the potential is $(1-\cos\phi)$ and so the field is in its vacuum state if it is equal $2n\pi$ for $n \in \mathbf{Z}$. In other words, there is a different degenerate vacuum for each n.

Obviously, a finite energy solution must approach the vacuum as $|x|$ becomes very large. The interesting thing is that the field may approach a different vacuum in each of the two spatial directions. If this is the case, the field may not be in the vacuum everywhere, it must interpolate between the two vacuum values. This means the field must have some potential energy: it is not in a zero energy state. However, there is a barrier between a state of this type, with different vacua at the positive and negative spatial infinities, and the vacuum itself. No dynamical process may occur which changes between these two possibilities. This is because changing the field so that it is in the same vacuum state everywhere requires that the field moves up and over the potential barrier all the way out to infinity in one of the two directions. This takes an infinite amount of energy. The difference in the vacuum state as x goes to positive and negative infinite is $2k\pi$ where $k \in Z$. This integer, k, is preserved by field evolution.

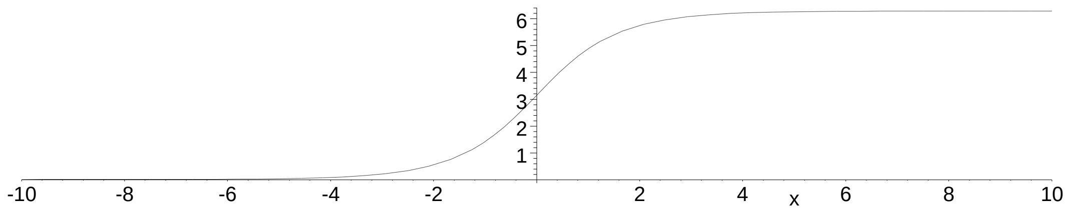
In fact this integer can be considered to be a winding number.[3] The Lagrangian only depends on the value of the field modulo 2π and so the field can be considered to be an angle: a U_1-valued field. If this is the case, then the finite energy condition requiring that

[1] The sine-Gordon model was invented by Tony Skyrme, the name is a joke because it sounds like Klein-Gordon. The word soliton was invented by Martin Kruskal, though solitary waves themselves were discovered in the first half of the nineteenth century by someone called Scott Rusell.

[2] If you don't have any physics, don't worry too much about this example, a bosonic field theory in 1+1-dimensions is a dynamical system for a field, that is a function of one space dimension and time. A classical solution is one that minimizes the action for some Lagrangian L which is itself the space integral of a Lagrangian density $\mathcal{L}$. In quantum field theory, particles correspond to fluctuations of this space-time function.

[3] In certain theories, including this one, it can also be considered to be a central charge, see the celebrated paper by Witten and Olive.

(i)



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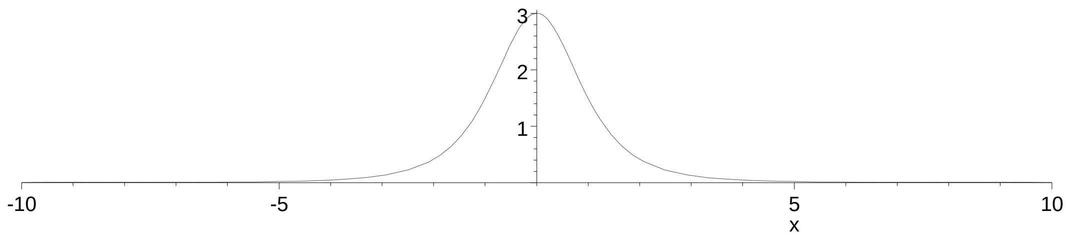

Figure 1: Plots of (i) the field and (ii) the energy of the static kink.

the field take a vacuum value at spatial infinity becomes a requirement that the field takes the same value, zero, at both negative and positive spatial infinity. This means that the field at a given time can be considered as a map from S^1 to S^1.[4] This map has a winding number and this winding number the integer mentioned above.

There is a known[5] static solution to the sine-Gordon model. It is

$$\phi = 4 \arctan e^{x}. \tag{2}$$

This solution is known as the **kink**. It is plotted in Fig. 1. It has winding number one. The solution with winding number minus one

$$\phi = 4 \arctan e^{-x} \tag{3}$$

is called an **anti-kink**. The kink is an exponentially localized lump of energy centered around $x = 0$.[6] Other static solutions can be obtained by translating this solution and moving solutions can be obtained by Lorenz transform. Two of these kinks at different positions can be superimposed and allowed to evolve, they will move away from each other.

[4]Space is not topologically S^1; because the field has the same value at both $x = -\infty$ and $x = \infty$ the two ends of the real line can be considered as being joined up and so with this boundary condition it is possible to add a point at infinity compactifying space.

[5]Don't fail to be surprised by this, the Euler-Lagrange equations are non-linear, it is not surprising that there is a solution but it is surprising that it is known analytically. This is very unusual and only occurs because the sine-Gordon model is **integrable**: there is an infinite number of conserved quantities.

[6]Which can be considered to be the position of the kink.

In many ways a kink behaves like a particle[7] and is an example of what is called a soliton.[8]

In the quantized sine-Gordon model the field quanta behave like particles in the normal way, what isn't normal is that the solitons also behave like particles. This means there are two different particle spectra: a soliton spectrum and a spectrum arising out of quantization. There is another model called the Thirring model which also has two spectra and it was discovered that the two spectra match the two spectra of the sine-Gordon model, but only if they are switched. Thus, the soliton spectrum of the sine-Gordon model is the same as the spectrum of field quanta in the Thirring model and visa versa. This phenomena is surprising and deep and occurs elsewhere in field theory. It is similar to, and linked with, the T-duality discussed in the string theory example.

Triangulation and homotopy

Triangulation is a powerful and rigorous method of calculating homotopy. It is a useful idea which has applications beyond homotopy and will be used for defining homology. The idea behind triangulation is to simplify the manifold to something more finite: a collection of triangles, tetrahedrons and their higher dimensional generalizations. The generalization of a tetrahedron to any dimension is the m-**simplex**:[9]

$$\sigma^m = \{\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_{m+1} x_{m+1} | \lambda_i \geq 0, \sum_{i=1}^{m+1} \lambda_i = 1\} \tag{4}$$

Obviously the simplex depends on a set of distinct points $\{x_1, x_2, \ldots, x_{m+1}\}$, these are the corners of the simplex. A useful notation is to write $\langle x_1 x_2 \ldots x_{m+1} \rangle$ to denote the simplex defined by that set of points.

The subspace of the m-simplex defined by setting one of the λ_i's to zero is a $(m-1)$-simplex. This is a **face** of the m-simplex. In fact, it is convenient to define the k-face as the k-simplex derived by setting $m-k$ of the λ_i's to zero. Thus, a 3-simplex is homeomorphic to a tetrahedron, it has four 2-faces each of which is a triangle, or 2-simplex. The 3-simplex also has six 1-faces and each of these is a line segment, or 1-simplex. Finally, it has four 0-faces, each of these is a point, or 0-simplex.

A space of dimension n is **triangulable** if it is homeomorphic to a collection of n-simplices satisfying the condition that the intersection of two of the simplices is either

[7]Nice animations of kink-kink and kink-anti-kink interactions can be found on the web, the nicest I have seen is http://www.math.h.kyoto-u.ac.jp/ takasaki/soliton-lab/gallery/solitons/sg-e.html

[8]The kink cannot decay even if it is perturbed, this is guaranteed by the winding number. In fact, the sine-Gordon model is peculiar in that a kink-anti-kink configuration does not decay to the vacuum either, in short, if a kink is superimposed with an anti-kink at a different position they will move towards each other but they will not annihilate, they will pass through each other. In fact, there is an oscillating solution known as a **breather**. This solution has winding number zero but is stable, at least to small perturbations. Kink-anti-kink annihilation is allowed by the conservation of winding number and occurs in similar models. It does not occur in the sine-Gordon model because the sine-Gordon model is integrable.

[9]That is, an m-dimensional simplex

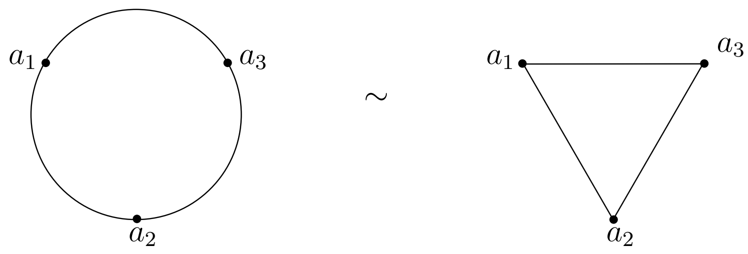

Figure 2: The triangulation of a circle.

empty or itself a simplex.[10] This means that if two simplices intersect they intersect along a single shared face. The idea here is to calculate a fundamental group of the triangulable space[11] by calculating the fundamental group of the corresponding collection of n-simplices. This is a useful method because we have replaced something which is differentiable by something finite: the triangulation.

The collection of n-simplices making up the triangulation along with all the faces of those simplices is called a **simplicial complex**; more generally, a simplicial complex, K, is a set of simplices satisfying the intersection property above along with the closure condition that the face of any simplex in K is also in K.[12]

The triangluation of a circle is given in Fig. 2. The corresponding simplicial complex is

$$K = \{\langle a_1 a_2\rangle, \langle a_2 a_3\rangle, \langle a_3 a_1\rangle, \langle a_1\rangle, \langle a_2\rangle, \langle a_3\rangle\}. \tag{5}$$

The set $\{\langle a_1 a_2\rangle, \langle a_2 a_1\rangle, \langle a_1\rangle, \langle a_2\rangle\}$ is not a simplicial complex because the intersection of $\langle a_1, a_2\rangle$ and $\langle a_2, a_1\rangle$ is not a simplex. If the circle is replaced by a disc the triangulation corresponds to the complex

$$K = \{\langle a_1 a_2 a_3\rangle, \langle a_1 a_2\rangle, \langle a_2 a_3\rangle, \langle a_3 a_1\rangle, \langle a_1\rangle, \langle a_2\rangle, \langle a_3\rangle\}. \tag{6}$$

A triangulation of the 2-torus is given in Fig. 3. It is surprisingly complicated, but anything simpler does not satisfy the intersection rule.[13] An example of something simpler which does not satisfy the triangulation rule is given in Fig. 4.

Since a triangulable manifold is homeomorphic to its triangulation, its fundemental group can be calculated by calculating the fundemental group of the triangulation. This is less troublesome than it sounds because the fundemental group of a triangulation is isomorphic to another group called the **edge group**. The edge group is a group generated by the 1-simplices of a triangulation, modulo certain relations which will be described

[10]That is, if $\sigma_1 \cap \sigma_2$ is n-dimension it is an n-simplex.

[11]According to Spivak it is a difficult theorem that every differentiable manifold has a triangulation.

[12]Throughout this discussion I am slightly vague about what I mean by a triangulation, do I mean the the homeomorphic set of n-simplices or do I mean the corresponding simplicial complex. In fact, it should be clear by context.

[13]Simpler is a subjective term, there is a triangulation of the torus in Nash and Sen with fewer triangles, however, it is harder to draw. I saw this one in Nakahara.

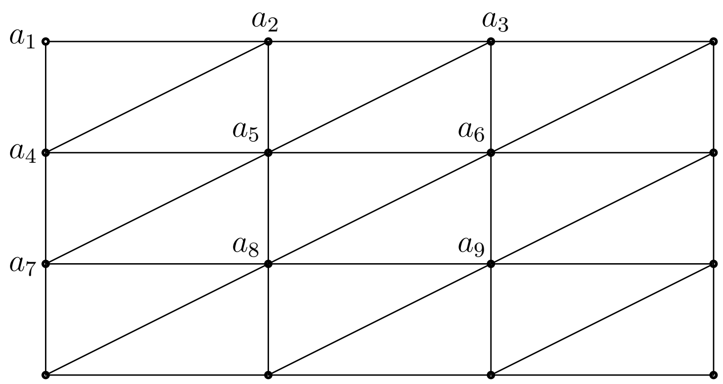

Figure 3: The triangulation of a torus. Remember, the opposite sides are identified and so there are no unlabelled points.

below. The isomorphism itself is apparently to be found in a book called *Basic Topology* by M.A. Armstrong, a copy of which it is on its way from Santry. However, a nice explaination of how the proof works is given in Nash and Sen and repeated below.

Inside any triangluation it is possible to find a maximal **tree**. A **tree** is a connected set of 1-simplices with no loops: it looks like a tree.[14] A tree inside a triangulation is maximal if it is not possible to add another 1-simplex without creating a loop. Clearly, a maximal tree contains all the 0-simplices in the triangulation.

To obtain the edge group of a triangulation,[15] K, a maximal tree, T, is chosen and a group generator g_{ij} is assosiated to each ordered 1-simplex $\langle a_i a_j \rangle$ with the opposite ordering giving the inverse: $g_{ij} = g_{ji}^{-1}$. To form the edge group, generators assosiated to a 1-simplex in T are set equal to the idenity and the 2-simplex relation is imposed: if $\langle a_1 a_2 a_3 \rangle \in K$ then $g_{12}g_{23}g_{31} = \mathbf{1}$. Thus, the edge group is generated by edges in K which are not in T modulo the 2-simplex relation.

In the edge group there is a generator assosiated with each edge that is not in the tree. Since the tree is maximal each edge in $K \backslash T$ defines a loop and so the edge group assosiates a generator to a loop. It seems clear that every loop is homotopic to a loop of this sort: one that runs along the 1-simplices. However, there may be homotopies between different loops of this sort, this is accounted for by the 2-simplex relation.

This process of calculating edge groups becomes clear by considering examples. In the case of the circle in Fig. 5, $g_{13} = \mathbf{1}$ and $g_{12} = \mathbf{1}$ because both of these are in the tree. Thus, the edge group is the group generated by g_{23}. There are no further relations because there are no 2-simplices in the triangulation. The group with a single generator and no further

[14]In winter.

[15]The edge group is calculated using the 2-simplices and 1-simplices inside the simplicial complex corresponding to a triangulation, since the spaces considered are all two dimensional, I haven't paid much attention to this distinction.

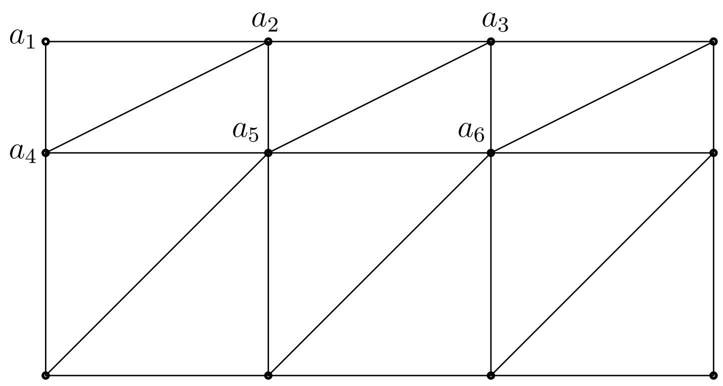

Figure 4: This is not a triangulation of a torus: note, for example, that $\langle a_1a_2a_4\rangle \cap \langle a_4a_5a_1\rangle = \{a_1, a_4\}$ and that is not a simplex.

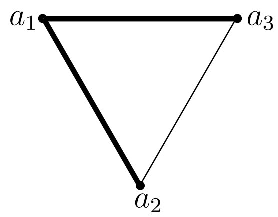

Figure 5: The triangulation of the circle with the maximal tree marked by thick lines.

relations is $\mathbf{Z}$. This is what we might have expected. The triangulation of the disc is the same, but with the addition of the 2-simplex $\langle a_1a_2a_3\rangle$. This adds a relation $g_{12}g_{23}g_{31} = \mathbf{1}$. Substituting $g_{13} = \mathbf{1}$ and $g_{12} = \mathbf{1}$ into this relation, we find $g_{23} = \mathbf{1}$ and so the edge group is trivial, again, as we might have expected.

The torus is a more complicated example. Refering to Fig. 6 and starting from the top righthand corner, $g_{13}g_{61} = \mathbf{1}$ so writing $h = g_{13}$ then $h = g_{16}$ also. The next 2-simplex is $g_{46}h^{-1} = \mathbf{1}$ so $g_{46} = h$ as well. In fact, moving down the righthand column $g_{49} = g_{79} = h$. Starting at the bottom left, $g_{17}g_{81} = \mathbf{1}$ so $k := g_{17} = g_{18}$. Moving along the bottom column gives $k = g_{28} = g_{29} = g_{39}$. That uses up all the relations except the two 2-simplices on the bottom right corner. The $\langle a_3a_7a_9\rangle$ 2-simplex gives $kh^{-1}g_{73} = \mathbf{1}$ and the $\langle a_1a_3a_7\rangle$ 2-simplex gives $hg_{37}k^{-1} = \mathbf{1}$. Eliminating g_{37} gives $kh = hk$ and so the group is $\mathbf{Z} \oplus \mathbf{Z}$.

The projective plane is easier than you might fear. From the $\langle a_1a_4a_5\rangle$ 2-simplex $g := g_{14} = g_{15}$, from the 2-simplices to each side $g = g_{24}$ and $g = g_{35}$ and from the $\langle a_2a_3a_4\rangle$ 2-simplex $g = g_{23}$. Finally the $\langle a_2a_3a_5\rangle$ 2-simplex gives $g = g_{32}$. This means $g^2 = \mathbf{1}$ and so the group is $\mathbf{Z}_2$.

The fundemantal group of the plane with two punctures is the non-Abelian group with

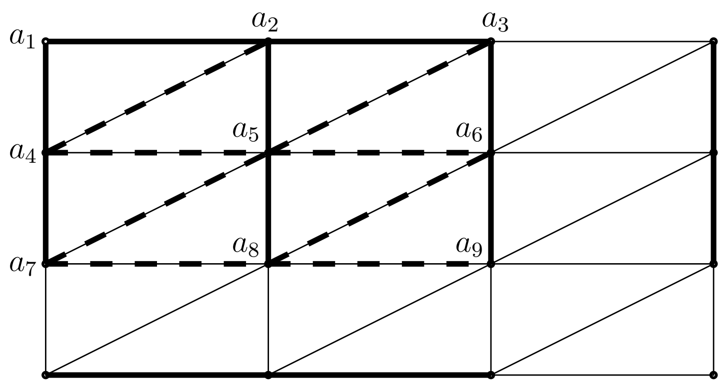

Figure 6: The maximal tree is marked by the solid thick line. Remember that opposite sides are identified so it is connected. If two edges on a 2-simplex have trivial generators, for example, if they are in the maximal tree, the 2-simplex relation means the generator corresponding to the third edge is also trivial. Edges of this sort are marked with dashed lines.

two generators and no relations. The triangulation of the twice punctured plane is given in Fig. 8. The two generators are g_{13} and g_{24} and since there are no 2-simplices there are no relations between them.

The obvious example that I've left out is S^2, this is an easy example, the triangulation is a tetrahedron. the Klein bottle is done in Nash and Sen and is quite entertaining. It is quite like the torus example, obviously.

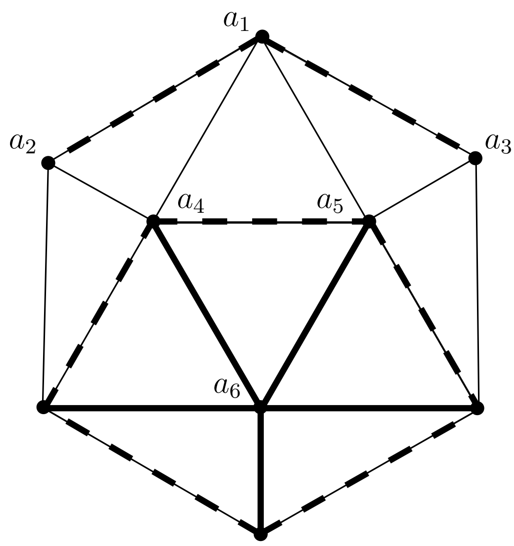

Figure 7: A triangulation of $\mathbf{RP}^2$ with the maximal tree marked along with the edges that are obviously trivial. Remember that opposite points are idenitified.

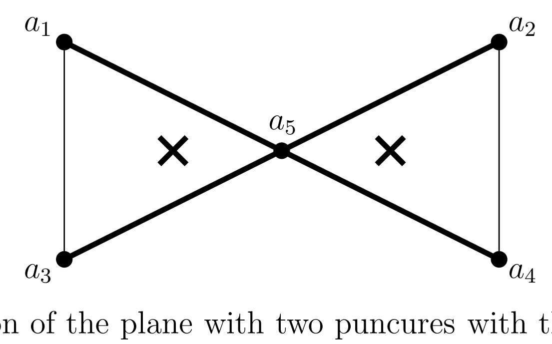

Figure 8: A triangulation of the plane with two puncures with the maximal tree marked. The punctures are marked with crosses