Rough notes for Maths 543

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Lecture 1

A useful place to begin a course on Topology and Geometry in Physics is the matrix group SU_2. This is the group of special unitary 2×2 matrices, in other words $U \in \mathrm{SU}_2$ if $U^\dagger U = \mathbf{1}_2$ and $\det U = 1$. It is easy to verify that this is a group under the normal matrix multiplication. The interesting thing is that the group is parameterized by the three-sphere, S^3.

A general 2×2 matrix has the form

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{1}$$

Since $U^\dagger = U^{-1}$, $\alpha = \bar{\delta}$ and $\beta = -\bar{\gamma}$. Thus, a general SU_2 matrix has the form

$$U = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \tag{2}$$

subject to the unit determinant constraint $|\alpha|^2 + |\beta|^2 = 1$. Writing $\alpha = a_1 + ia_2$ and $\beta = b_1 + ib_2$, SU_2 is parameterized by a point on the three-sphere

$$a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1. \tag{3}$$

This matrix group is an example of a Lie group. A **Lie group** is a space which is both a group and a smooth manifold. A manifold, in turn, is a space which locally looks like $\mathbf{R}^n$ for some n, thus it is covered by neighbourhoods each of which is homeomorphic to $\mathbf{R}^n$. A good example is the two-sphere. Near the north pole, the usual polar and azimuthal angles map the two-sphere to $\mathbf{R}^2$ in an obvious way

$$f_1 : (\theta, \phi) \mapsto (\theta \cos\phi, \theta \sin\phi) \tag{4}$$

but the map f_1 ceases to be one-one at $\theta = \pi$: f_1 maps the south pole to the whole of the circle of radius π. The map f_1 is a homeomorphism from $S^2\backslash\{s\}$ to the open disc of radius π in $\mathbf{R}^2$.[1] Another map from the two-sphere to $\mathbf{R}^2$ is given by

$$f_2 : (\theta, \phi) \mapsto ((\pi - \theta)\cos\phi, (\pi - \theta)\sin\phi) \tag{5}$$

This map is a homeomorphism over all of $S^2 \setminus \{n\}$. In this way, the two-sphere can be covered with two open sets, $S^2 \setminus \{s\}$ and $S^2 \setminus \{n\}$ and on each of these open sets there is

[1] $S^2 \setminus \{s\}$ is the two-sphere minus the point s, the south pole. The open disc of radius 2π in $\mathbf{R}^2$ might be called $B^2_{2\pi}$ and for general radius is $B^2_r = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 < r^2\}$ The reason to write B not D is that the open disc is the two-dimensional open ball the same way the circle is a one-sphere and is called S^1. The closed ball is $\bar{B}^2_r = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 \leq r^2\}$.

a homeomorphism onto an open subset of $\mathbf{R}^2$. This means that locally the two-sphere is homeomorphic to $\mathbf{R}^2$ but globally it is not.

A neighbourhood along with a map from the neighbourhood onto a neighbourhood in $\mathbf{R}^n$ is a **coordinate patch**. The existence of these coordinate patches is the defining property of a manifold. The local coordinatization also allows properties like smoothness or analyticity to be defined. In the two-sphere example above the local coordinatization gives a map between punctured discs, a point in $S^2 \setminus \{n, s\}$ may be mapped to the punctured disc $B^2_{2\pi} \setminus (0, 0)$ by either f_1 or f_2. Since f_1 and f_2 are invertible this means there is an invertible map

$$f_1 f_2^{-1} : B^2_\pi \setminus (0, 0) \to B^2_\pi \setminus (0, 0) \tag{6}$$

which is given by

$$f_1 f_2^{-1} : (x, y) \mapsto ((\pi/r - 1)x, (\pi/r - 1)y) \tag{7}$$

and is a smooth map. This shows the two-sphere is a smooth manifold.

These ideas are made definite by some definition. A **n-dimensional differentiable manifold**,[2] M, is a topological space together with a set of pairs (U_α, f_α) where the U_α are open subsets of M such that $\cup_{\alpha \in A} U_\alpha = M$ and each f_α is a homeomorphism onto an open subset of $\mathbf{R}^n$. Furthermore, if the intersection of U_α and U_β is none empty $f_\alpha f_\beta^{-1}$ is infinitely differentiable. The open subsets are sometimes called **coordinate charts**, or just charts and the maps are called **coordinate maps**.[3]

Before going on to examine examples, it should be noted that there are two parts to the definition, first, the topological space is required to be locally homeomorphic to $\mathbf{R}^n$ and secondly this local homeomorphism is used to require that there is a differential structure. The second condition can also be stregthned to give an analytic manifold or, though this never happens in real life, it could be weakened to give a C^s manifold for some $0 < s < \infty$. The second condition also allows us to define a **differentiable function**: this is a map between manifolds which is a differential map when extended by the coodinate maps to give a map between $\mathbf{R}^n$ and $\mathbf{R}^m$. A **diffeomorphism** is a differential one-one map whose inverse is also differential.

Example 1 An easy example of this is the circle S^1. For convenience the circle is taken to have unit radius so that a point on the circle is $(\cos\theta, \sin\theta)$. There is an obvious map onto the real line given by

$$f_1 : (\cos\theta, \sin\theta) \mapsto \theta \tag{8}$$

however, this is only a single-valued function if the value of θ is restricted: $\theta = 0$ and $\theta = 2\pi$ correspond to the same point on the sphere. Thus, a point must be excised[4] from S^1 and the coordinate chart is $U_1 = S^1 \setminus \{\theta = 0\}$. Another coordinate chart can be found

[2]Differentiable, C^∞ and smooth are being used to mean the same thing.

[3]The index set A is mostly a notational convenience, usually there is a small number of charts (U_α, f_α), in the two-sphere example above $A = \{1, 2\}$. When defining diffeomorphisms between differential manifolds it is convenient to consider a maximal set of charts, in this case A becomes a much larger indexing set. This nicety won't concern us, but is well described in Spivak, for example.

[4]excise: 'to cut out from'

by using the same map but with a different range:

$$f_2 : (\cos\theta, \sin\theta) \mapsto \theta \tag{9}$$

and $U_2 = S^1 \setminus \{\theta = \pi\}$, in other words,

$$f_2 : U_2 \to (-\pi, \pi) \tag{10}$$

Now the the intersection is made up of two disjoint sets[5] $\{S^1|y > 0\}$ and $\{S^1|y < 0\}$. Thus the map $f_1 f_2^{-1}$ is made up of two parts

$$f_1 f_2^{-1} : (-\pi, 0) \cup (0, \pi) \to (0, \pi) \cup (\pi, 2\pi) \tag{11}$$

with

$$f_1 f_2^{-1} : \begin{cases} \theta \mapsto \theta & \theta \in \ (0, \pi) \\ \theta \mapsto 2\pi + \theta & \theta \in \ (-\pi, 0) \end{cases} \tag{12}$$

Clearly this is a differentiable map.

Example 2. Another obvious example is the two-torus $T^2 = S^1 \times S^1$. Since the torus is just a Cartesian product of circles, the coordinatization is like the coordinatization of the circle. In the torus case there are two angles instead of one and to ensure the map is single valued and continuous, two circles must be removed instead of a point.

Example 3. A useful example is the Riemann sphere, this might seem repetitive since the two-sphere was already treated as the preliminary example, however, finding the stereographic coordinates of a sphere is a useful thing to do. For definiteness we consider the sphere in $\mathbf{R}^3$ given by $x^2 + y^2 + z^2 = 1$ and we will stereographically project onto the plane $z = 0$. It is convenient to use the complex coordinate $\zeta = x + iy$ on this plane. The stereographic projection from the north pole of a point on the sphere is found by joining that point and the sphere by a straight line and continuing the line until it intersects the plane. The coordinate of that intersection point is the coordinate for the point on the sphere under stereographic projection. It is shown in Fig. (1) that the stereographic coordinate of the point $(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$ is

$$\zeta = e^{i\phi} \cot\frac{\theta}{2}. \tag{13}$$

The chart corresponding to this coordinate map is $S^2 \setminus \{n\}$, the north pole is projected to the point at infinity, which, of course, is not in $\mathbf{R}^2$. There is another coordinate map given by projection from the south pole, the stereographic coordinate for this projection is given by

$$\zeta' = e^{-i\phi} \tan\frac{\theta}{2} \tag{14}$$

[5]Don't let this confuse you, you just deal with each bit separately. Each bit is an open set so the gaps between them don't affect differentiability

and corresponds to the coordinate map $S^1 \setminus \{s\}$.[6] On the intersection of the two coordinate charts the two coordinates are related by

$$\zeta' = \frac{1}{\zeta}. \tag{15}$$

This is a differentiable relation.

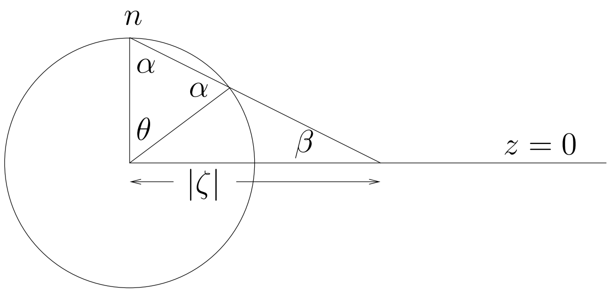

Figure 1: The steregraphic projection of the sphere from n. $|\zeta| = \cot\theta/2$ follows from $\theta + 2\alpha = \pi$ and $\alpha + \beta + \pi/2 = \pi$ and hence $\beta = \theta/2$.

Example 4. Next we look at **complex projective space**. The complex projective line, $\mathbf{CP}^1$, is the space of lines in two-dimensional complex space. It is the quotient space $\{(z_1, z_2) \in \mathbf{C}^2 | (z_1, z_2) \neq (0, 0)\}/\sim$ where $\sim$ denotes the relation $(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$ with $\lambda \in \mathbf{C}^\times$.[7] $z = z_1/z_2$ is a good coordinate for $\mathbf{CP}^1$ since it different for different equivalence classes and the same for every element of an equivalence class. It is a good coordinate for every point in $\mathbf{CP}^1$ except $[(1, 0)]$[8] Another coordinate is $z' = z_2/z_1$ and this is a good coordinate everywhere except $[(0, 1)]$. In the intersection, when neither z_1 nor z_2 is zero, the two coordinates are related by $z' = 1/z$. This demonstrates that the complex projective line and the Riemann sphere are identical manifolds. Of course the complex projective line can be generalized to n complex dimensions to give

$$\mathbf{CP}^n = ((z_1, z_2, \ldots, z_{n+1}) \in \mathbf{C}^{n+1} | (z_1, z_2, \ldots, z_n) \neq (0, 0, \ldots, 0)/\sim \tag{16}$$

where $\sim$ is the equivalence relation $(z_1, z_2, \ldots, z_{n+1}) \sim (\lambda z_1, \lambda z_2, \ldots, \lambda z_{n+1})$. Coordinates for a complex projective space are found the same way as the coordinates for the complex projective line: $(z_2/z_1, z_3/z_1, \ldots, z_{n+1}/z_1)$ are coordinates provided $z_1 \neq 0$, $(z_1/z_2, z_3/z_2, \ldots, z_{n+1}/z_2)$ are coordinates provided $z_2 \neq 0$ and so on.

[6] One possible confusion is that the phase of ζ' is the conjugate of the phase of ζ. The point is that the complex coordinate for the projection from the south pole is $x - iy$ because the projection is onto the $z = 0$ plane from below. This choice is justified by the result: a holomorphic relation between the two coordinates.

[7] $\mathbf{C}^\times$ means the none zero complex numbers, the idea being that multiplication is an invertible operation on $\mathbf{C}^\times$

[8] $[(1, 0)]$ is the equivalence class of $(z_1, z_2) = (1, 0)$, that is, it is the point in $\mathbf{CP}^1$ corresponding to the points in $\mathbf{C}^2$ of the form $(\lambda, 0)$.

Example 5 Real projective space is like complex projective space except everything is real rather than complex. Thus

$$\mathbf{RP}^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} | (x_1, x_2, \ldots, x_{n+1}) \neq (0, 0, \ldots, 0)\}/\sim \tag{17}$$

with $\sim$ being the equivalence relation $(x_1, x_2, \ldots, x_{n+1}) \sim (\lambda x_1, \lambda x_2, \ldots, \lambda x_{n+1})$ and this time $\lambda \in \mathbf{R}^\times$. Coordinates are found in just the same way that coordinates are found for $\mathbf{CP}^n$, the reason for introducing $\mathbf{RP}^n$ is not really to talk about coordinates again, but rather, to give an example of a space with non-trivial homotopy.[9]

The non-trivial homotopy is easy to understand if you first understand what the manifold looks like. For definiteness we consider the two-dimensional example of $\mathbf{RP}^2$. This is the space of lines through the origin in $\mathbf{R}^3$, two points are on the same line through the origin precisely when they are equivalent under $\sim$. Now, every point in $\mathbf{R}^3$ is equivalent under $\sim$ to a point on the sphere

$$x_1^2 + x_2^2 + x_3^2 = 1 \tag{18}$$

and so a point on the sphere represents an equivalency class in $\mathbf{R}^3$. However, requiring $x_1^2 + x_2^2 + x_3^2 = 1$ does not fix λ: on the sphere the two antipodal points (x_1, x_2, x_3) and $(-x_1, -x_2, -x_3)$ are equivalent. Thus

$$\mathbf{RP}^2 = S^2/\sim|_{S^2} \tag{19}$$

where $\sim|_{S^2}$ is the antipodal map. This is all a way of saying that the space of lines through the origin is the same as the space of directions in space quotiented by the identification of opposite directions.

It is convenient to remove many of the repeated points, by replacing the sphere by the hemisphere with $x_3 \geq 0$. It should be noted though that the equivalence relation still acts on the circle $x_3 = 0$. Finally, under projection onto the x_1x_2-plane the closed hemisphere with $x^3 \geq 0$ is homeomorphic to the closed ball $\bar{B}_1^2$. Thus $\mathbf{RP}^2$ is homeomorphic to the closed ball with points on the boundary identified. In other words,

$$\mathbf{RP}^2 \equiv \{(x_1, x_2) \in \mathbf{R}^2 | x_1^2 + x_2^2 \leq 1, \text{antipodal points on the boundary identified}\}. \tag{20}$$

With this model for $\mathbf{RP}^2$ it is easy to see that it has non-trivial homotopy. Consider a loop in $\mathbf{RP}^2$, if the loop doesn't touch the boundary it can be continuously deformed to a point. If it does touches the boundary at one pair of antipodal point, it can't be deformed to a point. This is illustrated in Fig. 2. A loop that touches the boundary at two pairs of antipodal points can be deformed to a point, this is illustrated in Fig.3. Thus it is in the same class under deformation as the loop that never touches the boundary, but the loop that touches the boundary once is in a separate class. Two loops that can be deformed into each other are called homotopic loops and a group whose elements are homotopy classes of loops may be defined. This group is called the fundamental group.

[9] I haven't said what homotopy is yet, the idea is to give an example, then explain what it is and then define it.

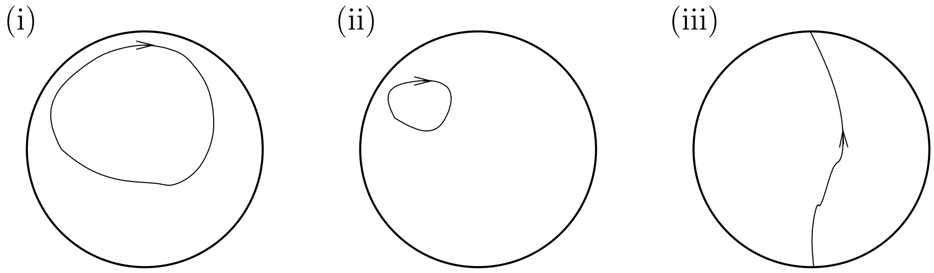

Figure 2: The loops in (i) and (ii) can be deformed into each other and indeed can be deformed continuously to a point. However, the loop in (iii) is of a different type and cannot be deformed continuously to a point.

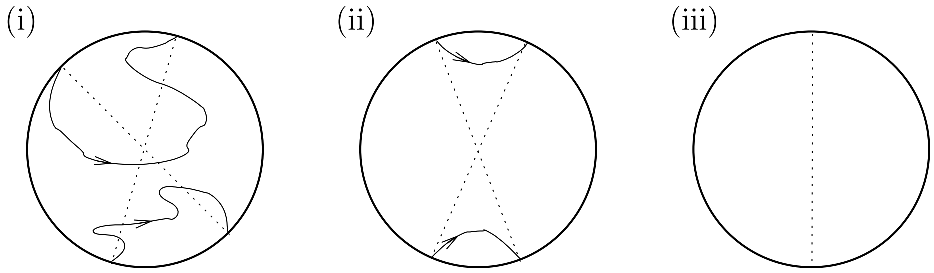

Figure 3: A loop which touches the boundary at two different points can be deformed to a point.

Before giving a more formal definition of homotopy and defining the group structure on the fundamental group, it is interesting to find that the group manifold of SO_3, the group of rotations, is $\mathbf{RP}^3$ and the non-triviality of its fundamental group can be easily observed using a cup and your arm.

SO_3 is the group of special orthogonal 3×3 matrices and is obviously the group of rotations because, if $R \in SO_3$ and $\mathbf{v} \in \mathbf{R}^3$ then $|R\mathbf{v}|^2 = \mathbf{v}^T R^T R\mathbf{v} = \mathbf{v}^T\mathbf{v} = |\mathbf{v}|^2$. In the same way that $\mathbf{RP}^2$ is homeomorphic to the closed two-ball with an antipodal identification on the boundary, $\mathbf{RP}^3$ is homeomorphic to the closed three-ball with antipodal points on the bounding two-sphere identified. A point in this three-ball can written in the form $(\theta/2\pi)\mathbf{n}$ where $\mathbf{n}$ is a unit vector. The correspondence between SO_3 and $\mathbf{RP}^3$ can be given by mapping this point to a clockwise rotation by θ around the line given by $\mathbf{n}$. Since a clockwise rotation of π around $\mathbf{n}$ is the same as a clockwise rotation of $-\pi$ around $\mathbf{n}$ this mapping is consistent with the identification of points on the boundary.[10]

The loops in $\mathbf{RP}^3$ fall into two homotopy classes, the same as for $\mathbf{RP}^2$. This means

[10]Explicitly, a rotation by π about $\mathbf{n} = (n_1, n_2, n_3)$ is given by the matrix

$$R_{ij}(\mathbf{n}, \theta) = \epsilon_{ijk} n_k \sin\theta + (\delta_{ij} - n_i n_j)\cos\theta + n_i n_j.$$

that there are two types of loops in SO_3. This is what is demonstrated using the cup.

In fact, since $\mathbf{RP}^3 = S^3/\sim$ there is a map from SU_2 to SO_3 which identifies antipodal points in SU_2. Explicitely this map is

$$R(U)_{ij} = \frac{1}{2}\mathrm{trace}\,(\sigma_i U \sigma_j U^{-1}) \tag{21}$$

and U and $-U$ have the same image in SO_3.[11] In fact, SU_2 and SO_3 have the same local group structure[12] but the have different topology: there is only one homotopy class of loops in SU_2, there are two homotopy classes in SO_3. The striking thing is that electrons and other fermions, represent the group SU_2 not SO_3, there is a celebrated discussion of this phenomena given by Feynman in his Dirac memorial lecture.

[11]If U is written

$$U = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\mathbf{n}\cdot\boldsymbol{\sigma}$$

then the double covering map recovers the parameterization of SO_3 given in the previous footnote.

[12]Or rather, the have the same Lie algebra.